CE 230A Notes

FALL 1963

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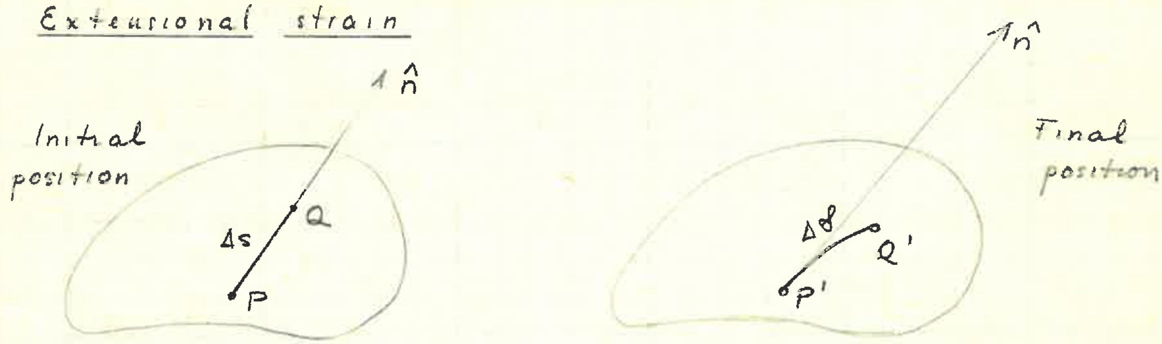
Notion of Deformation and Motion

The notion of motion is a question of relativity, it will depend on the position of the observer

Extensional strain



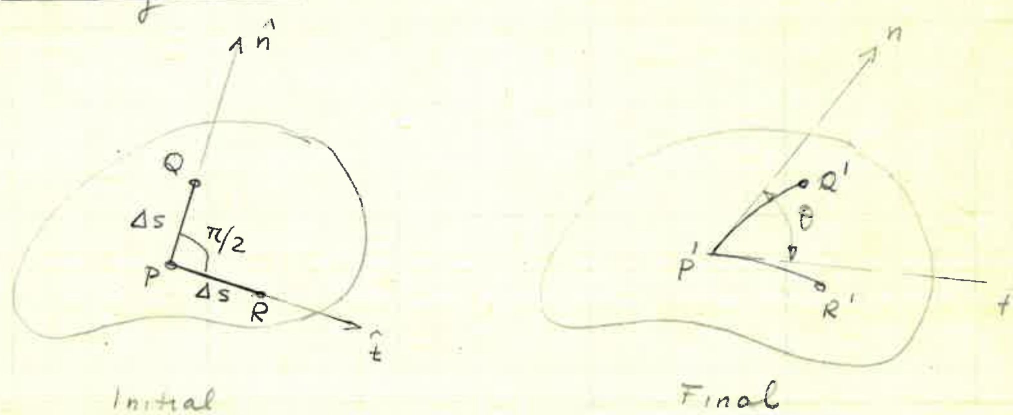
$\hat{n}$ is a direction <u>unit</u> vector

The extensional strain of the segment PQ is denoted by $\epsilon_n(P)$

$$\epsilon_n(P) = \lim_{Q \to P} \frac{\Delta\mathcal{S} - \Delta s}{\Delta s}$$

ϵ_n depends on
- point in body (P)
- direction $\hat{n}$
- magnitude of the limit

Shearing strain



The shearing strain of segments PQ, PR is denoted by γ_{nt}

$$\gamma_{nt}(P) = \lim_{\substack{Q \to P \\ R \to P}} \left(\frac{\pi}{2} - \theta \right)$$

depends on
- point P
- both $\hat{n}$ and $\hat{t}$
- magnitude of the limit

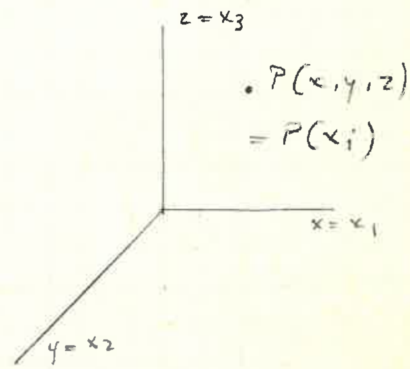
It is in general impossible to try to describe ϵ, γ for each point. Usually they are represented by function of point and directions

Reference: 'Th. of Elasticity' Timoshenko & Goodier
Def. of strain - displacements and strain - displacement equations

Index Notation. Summation Convention

A point $P(x, y, z)$ in a R.C.C. system can be represented by the shorter notation

$$P(x_i) \quad i = 1, 2, 3$$



x_i means x_1, x_2, x_3.

A linear algebraic equation shall be written as

$$\sum_{j=1}^{3} a_{ij} x_j = b_i \qquad i = 1, 2, 3,$$

where a_{ij} and b_i are constants. This means:

$$\begin{cases} i=1 & \sum_{j=1}^{3} a_{1j} x_j = b_1 \rightarrow a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ i=2 & \sum_{j=1}^{3} a_{2j} x_j = b_2 \rightarrow a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \\ i=3 & \sum_{j=1}^{3} a_{3j} x_j = b_3 \rightarrow a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \end{cases}$$

An even shorter notation is

$$a_{ij} x_j = b_i \qquad \text{cancelling } \Sigma \text{ symbol}$$

This is the <u>summation convention</u>: an index repeated in a expression is understood to be summed over the stated range

Example: $a_{jj} \qquad j = 1, 2, 3$

$$\longrightarrow a_{11} + a_{22} + a_{33}$$

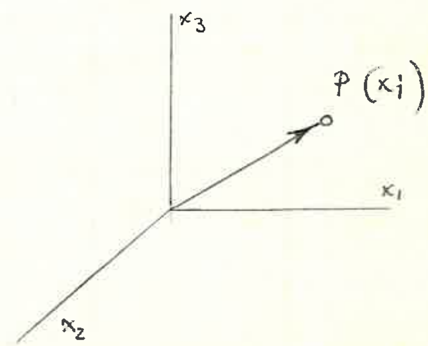
In a expression like $a_{ij} x_j$

i is said to be the <u>free</u> index (<u>can</u> take any value 1, 2, 3)
j " " <u>dummy</u> " (<u>must</u> " " " 1, 2, 3
i.e. <u>must</u> be summed over the entire range)

References on Cartesian Vector and Tensors

LASS, H	Vector and Tensor Analysis
COBURN, N	" " " "
ARIS, R	Vectors and Tensors
LONG, R	Mechanics of Solids and Fluids
HAWKINS, G	Multilinear Analysis

Position of a Point



The function of a point $P(x_i)$ is written using the index notation as $f(x_i)$

Some important functions $f(x_i)$ are

1. **Linear form**

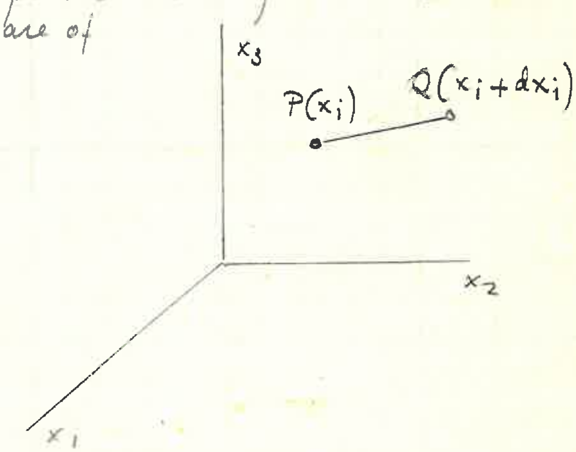
$$f(x_i) = a_1 x_1 + a_2 x_2 + a_3 x_3 =$$
$$= \sum_{i=1}^{3} a_i x_i = \boxed{a_i x_i}$$

2. Bilinear form (Quadratic form)

$$f(x_i, x_j) = a_{11} x_1 x_1 + a_{12} x_1 x_2 + \ldots$$
$$= \sum_{i=1}^{3} \sum_{j=1}^{3} a_{ij} x_i x_j = \boxed{a_{ij} x_i x_j}$$

3. Differential Quadratic Form

$a_{ij}\, dx_i\, dx_j$ represents the quadratic form of the distance between two neighboring points in a general curvilinear system. In a R.C.C., the square of the distance is



$$ds^2 = (dx_1)^2 + (dx_2)^2 + (dx_3)^2$$

corresponding to the matrix

$$a_{ij} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

row column

Total differential of $f(x_i)$

$$df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \frac{\partial f}{\partial x_3} dx_3 = \frac{\partial f}{\partial x_i} dx_i$$

introducing a new notation : $\frac{\partial f}{\partial x_i} = f,_i$

then $df = f,_i \, dx_i$

comma denotes partial differentiation with respect to the variable that follows, so

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = f,_{ij}$$

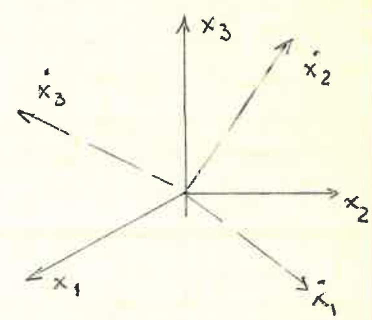
$$\frac{\partial^2 f}{\partial x_i \partial x_i} \leftarrow \cancel{\frac{\partial^2 f}{\partial x_i^2}} = f,_{ii} = \nabla^2 f. \quad \text{Laplacian of } f.$$

Chain Rule for differentiation

Let $f(x_i)$ given, while $x_i = x_i(\dot{x}_j)$

Form of f is given related to x_i R.C.C, while x_i coordinates are related to another system $\dot{x}_i$

$$\frac{\partial f}{\partial \dot{x}_j} = \frac{\partial f}{\partial x_i}\frac{\partial x_i}{\partial \dot{x}_j} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial \dot{x}_j} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial \dot{x}_j} + \frac{\partial f}{\partial x_3}\frac{\partial x_3}{\partial \dot{x}_j}$$



Let $f = \dot{x}_K$; Then in general rule

$$\frac{\partial \dot{x}_K}{\partial \dot{x}_j} = \frac{\partial \dot{x}_K}{\partial x_i}\frac{\partial x_i}{\partial \dot{x}_j}$$

if $j = K = 1, 2, 3$ $\qquad \frac{\partial \dot{x}_K}{\partial \dot{x}_j} = 1$

$j \neq K$ $\qquad \frac{\partial \dot{x}_K}{\partial \dot{x}_j} = 0$

If we define the Kronecker delta symbol δ_{jK} as

$$\delta_{jK} = \begin{cases} 0 & j \neq K \\ 1 & j = K \end{cases}$$

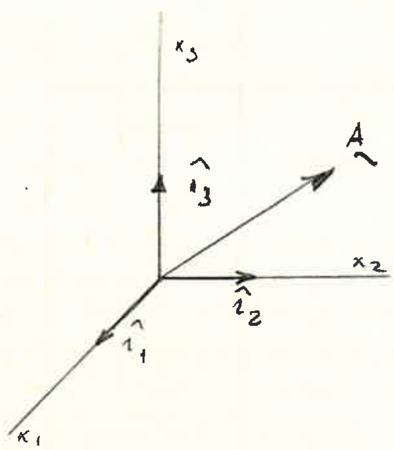
then

$$\boxed{\frac{\partial \dot{x}_k}{\partial x_i}\frac{\partial x_i}{\partial \dot{x}_j} = \delta_{jk}}$$

Example : meaning of δ_{ii}

$$\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 3$$

Base vectors , Coordinate transformations



$\hat{\imath}_1, \hat{\imath}_2, \hat{\imath}_3$ are base unit vectors

Any vector $\underset{\sim}{A}$ can be expressed as

$$\underset{\sim}{A} = A_1 \hat{\imath}_1 + A_2 \hat{\imath}_2 + A_3 \hat{\imath}_3$$

(A_1, A_2, A_3) components of $\underset{\sim}{A}$
(scalar)

Scalar Product

$$\underset{\sim}{A} \cdot \underset{\sim}{B} = |A|\,|B| \cos\theta$$

$|A|$ magnitude of A $\theta =$ angle $(\underset{\sim}{A}, \underset{\sim}{B})$ $|B|\cos\theta$ is the projection of $\underset{\sim}{B}$ over $\underset{\sim}{A}$, $\underset{\sim}{A} \cdot \underset{\sim}{B}$ is a scalar.

$$\underset{\sim}{A} \cdot \underset{\sim}{A} = |A|^2 \quad \text{norm}$$

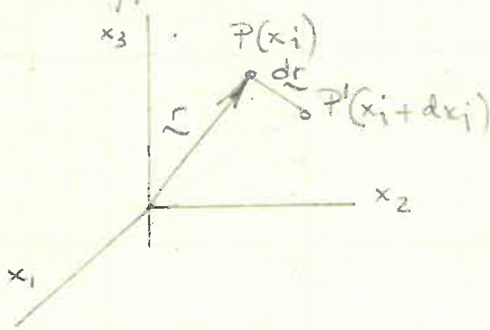
Among the units vectors $\hat{\imath}_K$:

$$\left.\begin{array}{ll} \hat{\imath}_1 \cdot \hat{\imath}_1 = 1 & \hat{\imath}_1 \cdot \hat{\imath}_2 = 0 \\ \hat{\imath}_2 \cdot \hat{\imath}_2 = 1 & \hat{\imath}_2 \cdot \hat{\imath}_3 = 0 \\ \hat{\imath}_3 \cdot \hat{\imath}_3 = 1 & \hat{\imath}_3 \cdot \hat{\imath}_1 = 0 \end{array}\right\} \quad \hat{\imath}_i \cdot \hat{\imath}_j = \delta_{ij}$$

If we write
$$\underset{\sim}{A} = A_1 \hat{\imath}_1 + A_2 \hat{\imath}_2 + A_3 \hat{\imath}_3 = A_K \hat{\imath}_K$$
$$\underset{\sim}{B} = B_1 \hat{\imath}_1 + B_2 \hat{\imath}_2 + B_3 \hat{\imath}_3 = B_j \hat{\imath}_j$$

$$\underset{\sim}{A} \cdot \underset{\sim}{B} = A_K B_j\, \hat{\imath}_K \hat{\imath}_j = A_K B_j \delta_{Kj} = \underline{A_K B_K}$$

Differential



$$\underset{\sim}{r} = \hat{\imath}_K x_K$$

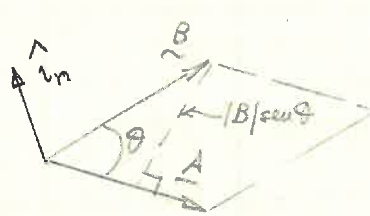
$$d\underset{\sim}{r} = \hat{\imath}_K dx_K \qquad (d\hat{\imath}_K = 0)$$ * unmovit

$$d\underset{\sim}{r}\, d\underset{\sim}{r} = ds^2 = (\hat{\imath}_K dx_K)(\hat{\imath}_j dx_j)$$

$$= \delta_{jK}\, dx_j\, dx_K = \underline{dx_j\, dx_j}$$

expression of the line element (RCC)

Cross Product



$$\underset{\sim}{A} \times \underset{\sim}{B} = |A|\,|B| \,\text{sen}\,\theta \cdot \hat{\imath}_n$$

$$\underset{\sim}{A} \times \underset{\sim}{B} = -\underset{\sim}{B} \times \underset{\sim}{A} \Leftarrow \text{right hand convention}$$

$|\underset{\sim}{A} \times \underset{\sim}{B}| =$ area parallelogram formed by $\underset{\sim}{A}$ and $\underset{\sim}{B}$

For the units vectors

$$\begin{array}{lll} \hat{\imath}_1 \times \hat{\imath}_1 = 0 & \hat{\imath}_1 \times \hat{\imath}_2 = \hat{\imath}_3 & \hat{\imath}_2 \times \hat{\imath}_1 = -\hat{\imath}_3 \\ \hat{\imath}_2 \times \hat{\imath}_2 = 0 & \hat{\imath}_2 \times \hat{\imath}_3 = \hat{\imath}_1 & \hat{\imath}_3 \times \hat{\imath}_2 = -\hat{\imath}_1 \\ \hat{\imath}_3 \times \hat{\imath}_3 = 0 & \hat{\imath}_3 \times \hat{\imath}_1 = \hat{\imath}_2 & \hat{\imath}_1 \times \hat{\imath}_3 = -\hat{\imath}_2 \end{array}$$

in condensed form $\hat{\imath}_j \times \hat{\imath}_K = e_{jKl} \cdot \hat{\imath}_l$ where /

$$e_{ijk} = \begin{cases} 0 & \text{if all indices are not distinct} \\ +1 & \text{if " " distinct and arranged in cyclic order} \\ -1 & \text{not cyclic order} \end{cases}$$

example $\hat{i}_1 \times \hat{i}_2 = e_{123} \cdot \hat{i}_3 = \hat{i}_3$

Triple scalar product*

$$[\underset{\sim}{A}, \underset{\sim}{B}, \underset{\sim}{C}] = \underset{\sim}{A} \cdot \underset{\sim}{B} \times \underset{\sim}{C} = \hat{i}_m A_m \cdot \hat{i}_\ell (B_j C_k e_{jk\ell}) =$$

$$= \delta_{m\ell} A_m B_j C_k e_{jk\ell} = A_\ell B_j C_k e_{jk\ell} = A_i B_j C_k \cdot e_{ijk}$$

Physical meaning = volume of prism $(\underset{\sim}{A}, \underset{\sim}{B}, \underset{\sim}{C})$

Gradient of a scalar

$f(x_i)$ scalar function

$$\text{Grad } f = \nabla f = \hat{i}_k f_{,k}$$

Vector operator "Del"**

$$\underset{\sim}{\nabla} = \hat{i}_1 \frac{\partial}{\partial x_1} + \dots = \hat{i}_k (\;)_{,k}$$

Divergence of a vector $\underset{\sim}{A}$

$$\text{div } \underset{\sim}{A} = \nabla \cdot \underset{\sim}{A} = \left(\hat{i}_k \frac{\partial}{\partial x_k}\right) \cdot \hat{i}_j A_j = A_{j,j}$$

Divergence of a gradient

$$\text{div grad } f = \nabla \cdot \nabla f = \nabla^2 f = f_{,kk}$$

$\nabla^2 f =$ Laplacian of f

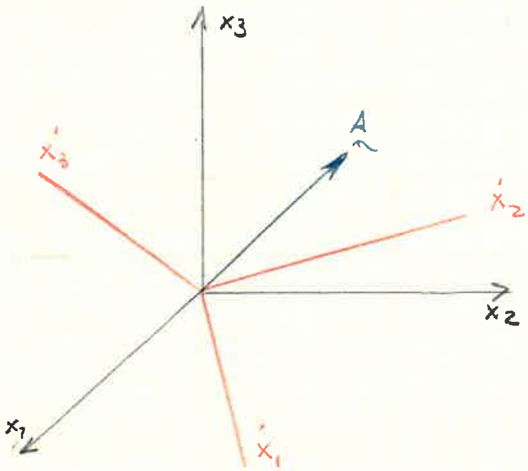
Curl*** of a vector $\underset{\sim}{A}$

$$\text{Curl } \underset{\sim}{A} = \nabla \times \underset{\sim}{A}$$

Invariant: expression that does not change its value after being subjected to a coordinate transformation.

* Producto triple   ** Nabla   *** Rotor

TRANSFORMATION OF COORDINATES



Vector $\underset{\sim}{A} = A_k \underset{\sim}{1_k}$

$$x_j = a_{jk} x_k' \qquad (j, k = 1, 2, 3)$$

$$x_1 = a_{11} x_1' + a_{12} x_2' + a_{13} x_3'$$

$$x_2 = a_{21} x_1' + \dots$$

$$x_3 = \dots$$

a_{jk} is called the <u>transformation matrix</u>. In this case (R.C.C.) each component has a immediate physical meaning: a_{jk} = director cosine of angle $(\underset{\sim}{x_j}, \underset{\sim}{x_k'})$

$$a_{jk} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ & a_{22} & a_{23} \\ & & a_{33} \end{pmatrix}$$

$$\frac{\partial x_j}{\partial x_\ell'} = a_{jk} \frac{\partial x_k'}{\partial x_\ell'} = a_{jk} \delta_{k\ell} = a_{j\ell} \qquad \longleftarrow 1$$

definition of vector.

note: it is necessary to use a free index ℓ. Of course, $\frac{\partial x_j}{\partial x_k} = a_{jk}$

The inverse transformation is

$$x_k' = a_{\ell k} x_\ell$$

$$\frac{\partial x_k'}{\partial x_\ell} = a_{\ell k} \qquad \longleftarrow 2$$

Multiply 1 and 2 and sum in k:

$$\frac{\partial x_j}{\partial x_k'} \frac{\partial x_k'}{\partial x_\ell} = a_{jk} a_{\ell k}$$

but the first member is also $\delta_{j\ell}$ Then

$$\boxed{a_{jk} a_{\ell k} = \delta_{j\ell}}$$

This particular type of linear transformation is taken as a <u>definition</u> of orthogonal transformation

Example $j = l = 1$ $a_{1K} a_{1K} = a_{11}^2 + a_{22}^2 + a_{33}^2 = 1$

$j = 1$ $l = 2$ $a_{1K} a_{2K} = a_{11} a_{21} + a_{12} a_{22} + a_{13} a_{23} = 0$

which are the well-known properties of director cosines.
The transformation of the unit base vectors are related by

$$\hat{i}'_j = a_{jK} \hat{i}_K$$

$$A = A_j \hat{i}'_j = A'_K \hat{i}_K = A_j a_{jK} \hat{i}_K = A_K \hat{i}_K \quad \therefore \rightarrow$$

$$(A'_K - a_{jK} A_j) \hat{i}_K = 0 \quad \rightarrow \quad \boxed{A'_K = a_{jK} A_j}$$

multiplying by a_{lK}:

$$a_{lK} A'_K = a_{lK} a_{jK} A_j = \delta_{jl} A_j = A_l \rightarrow$$

$$A_l = a_{lK} A'_K \quad \text{or} \quad \boxed{A_j = a_{jK} A'_K}$$

TENSORS

Let's be two vectors $\underset{\sim}{A}$ and $\underset{\sim}{B}$

$$A'_K = a_{jK} A_j \qquad B'_n = a_{ln} B_l$$

$$A'_K B'_n = a_{jK} a_{ln} A_j B_l$$

calling $C'_{Kn} = A'_K B'_n$ and $C_{jl} = A_j B_l$

$$\boxed{C'_{Kn} = a_{jK} a_{ln} C_{jl}}$$

definition of a tensor of order two (2 free indeces). A vector is a tensor of order one. For the fourth order it would be

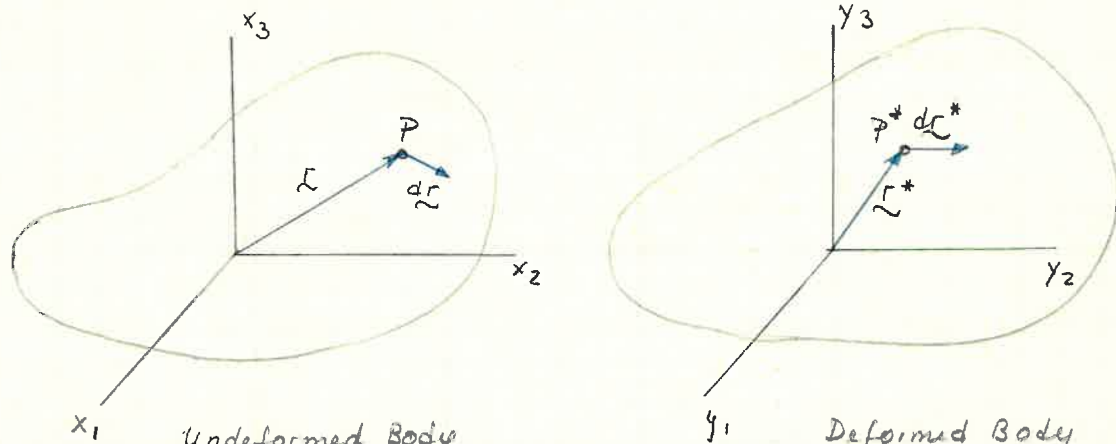
$$C'_{ijKl} = a_{mi} a_{nj} a_{pK} a_{ql} C_{mnpq}$$

In general tensors are function of position $C_{jl} = C_{jl}(x_i)$ $C'_{Kn} = C'_{Kn}(x'_i)$. Addition of substraction of tensors of the same order are performed simply by adding or substracting ~~the~~ respective components.

Contraction of a tensor: set two indeces equal and sum:

$j = K$ $A_{jj} = A_{11} + A_{22} + A_{33}$ scalar tensor of range zero

DEFORMATION MEASURES. STRAIN TENSORS



Undeformed Body at time = 0 | Deformed Body at time = t

$$ds = |d\underset{\sim}{r}| = |d\underset{\sim}{r} \cdot d\underset{\sim}{r}|^{1/2} \qquad ds^* = |d\underset{\sim}{r}^*| = |dr^* \cdot dr^*|^{1/2}$$

$$\boxed{ds^2 = dx_i \cdot dx_i} \qquad \boxed{ds^{*2} = dy_i\, dy_i}$$

y_i coordinates $\left\{\begin{matrix}\text{may be}\\ \text{are}\end{matrix}\right\}$ entirely independent of x_i

Spec. coord. transformation defining the deformation is

$$y_i = y_i\,(x_i) \qquad \underline{\text{not}} \text{ an orthogonal transformation}$$

we are to study the mapping of vector $d\underset{\sim}{r}$ into $d\underset{\sim}{r}^*$:

$$ds^{*2} = dy_i \cdot dy_i = \frac{\partial y_i}{\partial x_m} dx_m \cdot \frac{\partial y_j}{\partial x_n} dx_n =$$

$$= \frac{\partial y_j}{\partial x_m} \frac{\partial y_j}{\partial x_n} dx_m\, dx_n$$

$$ds^2 = dx_i\, dx_i = \delta_{mn}\, dx_m\, dx_n$$

Then

$$ds^{*2} - ds^2 = \left[\frac{\partial y_j}{\partial x_m}\frac{\partial y_j}{\partial x_n} - \delta_{mn}\right] dx_m\, dx_n =$$

$$= 2\, E_{mn}\, dx_m\, dx_n$$

where E_{mn} is the <u>Green (also St Venant) strain tensor</u>. The proof that E_{mn} is a symmetric tensor of range two is given in the exercise n° 3, set 2.

Because of the physical meaning $\frac{\partial y_j}{\partial x_m}\frac{\partial y_j}{\partial x_n} - \delta_{mn}$ is an invariant, what can be easily proofed directly, too.

Also $$C_{mn} = \delta_{mn} + 2E_{mn} = \frac{\partial y_j}{\partial x_m}\frac{\partial y_j}{\partial x_n}$$

is a tensor, called the <u>Green deformation tensor</u>, C_{mn} and E_{mn} is a Lagrangean tensor, referred to the original coordinate system.
If we change to another system x'_k in the undeformed state, then

$$E'_{ij} = a_{im}\,a_{jn}\,E_{mn} \quad , \text{etc.}$$

<u>Physical meaning of E_{mn}</u>

We define the extension ratio as $\lambda_{(n)} = \dfrac{ds^*}{ds_{(m)}}$

ratio of <u>final length</u> to the original <u>length</u> of a segment originally in the direction dr_m

$$\therefore \frac{ds^*}{ds_{(n)}} = \lambda_{(n)} = \sqrt{C_{mn}\frac{dx_m}{ds_{(n)}}\frac{dx_n}{ds_{(n)}}} =$$

$$= \sqrt{\frac{\partial y_j}{\partial x_m}\frac{\partial y_j}{\partial x_n}\frac{dx_m}{ds_{(n)}}\frac{dx_n}{ds_{(n)}}}$$

The extensional <u>strain</u> $\epsilon(n)$ is $\dfrac{ds^* - ds_{(n)}}{ds_{(n)}} = \lambda(n) - 1$

Note that $\dfrac{dx_m}{ds}$, $\dfrac{dx_n}{ds}$ are the components of the tangent vector to the space curve x_i. Then

$$\underset{\sim}{t} = \frac{d\underset{\sim}{x}}{ds}$$

$$\lambda(k) = \sqrt{\frac{\partial y_j}{\partial x_j}\frac{\partial y_j}{\partial x_k}\,n_j n_k}$$

Take $n_j = \cancel{n_i} = (1, 0, 0)$ directed along x_1 coordinate

$$\lambda(1) = \sqrt{\frac{\partial y_j}{\partial x_1}\frac{\partial y_j}{\partial x_1}} = \sqrt{C_{11}} = \sqrt{1 + 2E_{11}}$$

$$\epsilon(1) = \sqrt{1 + 2E_{11}} - 1$$

and $$\epsilon(2) = \sqrt{1 + 2E_{22}} - 1$$

$$\epsilon(3) = \sqrt{1 + 2E_{33}} - 1$$

DEFORMATION AND STRAIN

We had (page 11)

$$\epsilon(n) = \lambda(n) - 1 = \sqrt{C_{KL} \frac{dx_K}{ds_{(n)}} \frac{dx_L}{ds_{(n)}}} - 1$$

$dx_K/ds_{(n)}$ = unit vector tangent to axis $ds_{(n)}$. When $ds_{(n)}$ is taken along x_n coordinate axis $n_K = \delta_{Kn}$.

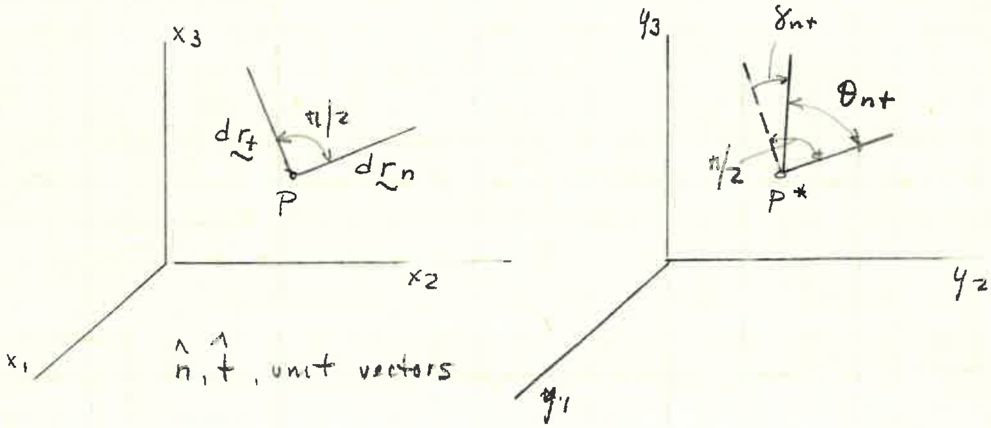
For instance

along x_1 $\quad n_K = (1,0,0) = \delta_{K1} \qquad \epsilon(1) = \sqrt{C_{11}} - 1 = \sqrt{1+2E_{11}} - 1$

$x_2 \quad (0,1,0) = \delta_{K2} \qquad \epsilon(2) = \sqrt{C_{22}} - 1 = \sqrt{1+2E_{22}} - 1$

$x_3 \quad (0,0,1) = \delta_{K2} \qquad \epsilon(3) = \sqrt{C_{33}} - 1 = \sqrt{1+2E_{33}} - 1$

For shear strains:



Shearing strain at $P' = \gamma_{nt}$

$$\cos\theta_{nt} = \frac{d\underset{\sim}{r}_n \cdot d\underset{\sim}{r}_t^*}{\sqrt{d\underset{\sim}{r}^* d\underset{\sim}{r}_t^*}}$$

$$\cos\theta_{nt} = \sin\gamma_{nt}$$

$$\sin\gamma_{nt} = \frac{dy_K^{(n)}\, dy_K^{(t)}}{\lambda_{(n)}\lambda_{(t)}\, ds_{(n)}\, ds_{(t)}}$$

but $$\frac{dy_K}{ds_{(n)}} = \frac{\partial y_K}{\partial x_i}\frac{\partial x_i}{ds_{(n)}} = \frac{\partial y_K}{\partial x_i} n_i$$

and $$\frac{dy_K}{ds_{(t)}} = \frac{\partial y_K}{\partial x_j}\frac{\partial x_j}{ds_{(t)}} = \frac{\partial y_K}{\partial x_j} n_j$$

$$\sin \gamma_{nt} = \frac{\frac{\partial y_k}{\partial x_i}\frac{\partial y_k}{\partial x_j} n_i n_j}{\lambda_{(n)}\,\lambda_{(t)}} = \frac{(\delta_{ij} + 2E_{ij})\, n_i n_j}{\lambda_{(n)}\,\lambda_{(t)}}$$

For instance

$$\sin \gamma_{12} = \frac{2E_{12}}{\sqrt{1+E_{11}}\,\sqrt{1+E_{22}}}$$

and similar expressions for $\sin\gamma_{23}$ and $\sin\gamma_{31}$

Case of small deformations

For a solid whose deformation are very small such that

$$\epsilon^2 << 1 \qquad \epsilon^2 \approx 0$$

$$\gamma^2 << 1 \qquad \gamma^2 \approx 0$$

then

$$[1+\epsilon(1)]^2 = 1 + 2\epsilon(1) + \cancel{\epsilon^2(1)}^{\,0} = 1 + 2E_{11}$$

$$\boxed{\epsilon_{11} \approx E_{11}}$$

$$\sqrt{1+2E_{11}}\,\sqrt{1+2E_{22}}\,\sin\gamma_{12} \cong [1+\epsilon(1)][1+\epsilon(2)]\,\gamma(12) \cong$$

$$\cong \gamma_{12} = 2E_{12}$$

$$\boxed{\gamma_{12} \approx 2E_{12}}$$

which characterize the small (or infinitesimal) deformation theory

Principal directions

The strain tensor at a point P

$$E_{ij} = \begin{vmatrix} E_{11} & E_{12} & E_{13} \\ & E_{22} & E_{23} \\ & & E_{33} \end{vmatrix}$$

can always be reduced to a diagonal tensor (i.e. one whose non-diagonal elements are zero) by a suitable change of coordinates. For these three directions n_j ($j = 1, 2, 3$) the strain vector is parallel to each of them.

Being E_j parallel to n_j

$$E_j = E \cdot n_j$$

where E is a scalar of proportionality.

$$E_i = \delta_{ij} \cdot E \cdot n_j = E_{ij} n_j$$

$$(E_{ij} - E\delta_{ij})\, n_j = 0 \qquad (i, j = 1, 2, 3)$$

To get a solution $n_{ij} \neq 0$ the determinant

$$\Delta = \det \left| E_{ij} - E\delta_{ij} \right| = 0$$

written in full

$$\begin{vmatrix} E_{11} - E & E_{12} & E_{13} \\ E_{12} & E_{22} - E & E_{23} \\ E_{13} & E_{23} & E_{33} - E \end{vmatrix} = 0$$

expanding $\quad E^3 - \vartheta_1 E^2 + \vartheta_2 E - \vartheta_3 = 0$

$$\vartheta_1 = E_{11} + E_{22} + E_{33} = E_{KK}$$

$$\vartheta_2 = E_{22}E_{33} + E_{33}E_{11} + E_{11}E_{22} - E_{13}^2 - E_{12}^2 - E_{23}^2 = \frac{1}{2}\delta^{i,j}_{m,n} E_{im} E_{jn}$$

$$\vartheta_3 = E_{11}E_{22}E_{33} + 2E_{12}E_{23}E_{31} - E_{11}E_{23}^2 - E_{22}E_{13}^2 - E_{33}E_{12}^2 =$$

$$= \frac{1}{6}\delta^{i,j,l}_{m,n,p} E_{1m} E_{jn} E_{lp}$$

$\vartheta_1, \vartheta_2, \vartheta_3$, because of the physical meaning of the cubic equation are invariants of the strain tensor. Note: $\vartheta_1, \vartheta_2, \vartheta_3$ form a basis and any other invariant (in general a invariant is any symmetric function of the elements of E_{ij}) can be expressed in function of them.

Solving the cubic, we get three real (because of the signs of coefficients) roots,

$$E_1 > E_2 > E_3$$

introducing them in the homogeneous system

$$(E_{ij} - E_p\delta_{ij})\, n_j^{(p)} = 0$$

we find $n_j^{(p)}$ as director coefficients of the principal axis, and the director cosines $\alpha_i^{(p)}$ by dividing by $\sqrt{n_i^{(p)} n_i^{(p)}}$.

In terms of the principal strains E_1, E_2, E_3 the invariants ϑ_1, ϑ_2 and ϑ_3 reduce to

$$\vartheta_1 = E_1 + E_2 + E_3 = \sum E_i$$

$$\vartheta_2 = E_1 E_2 + E_2 E_3 + E_3 E_1 = \sum E_i E_j$$

$$\vartheta_3 = E_1 E_2 E_3$$

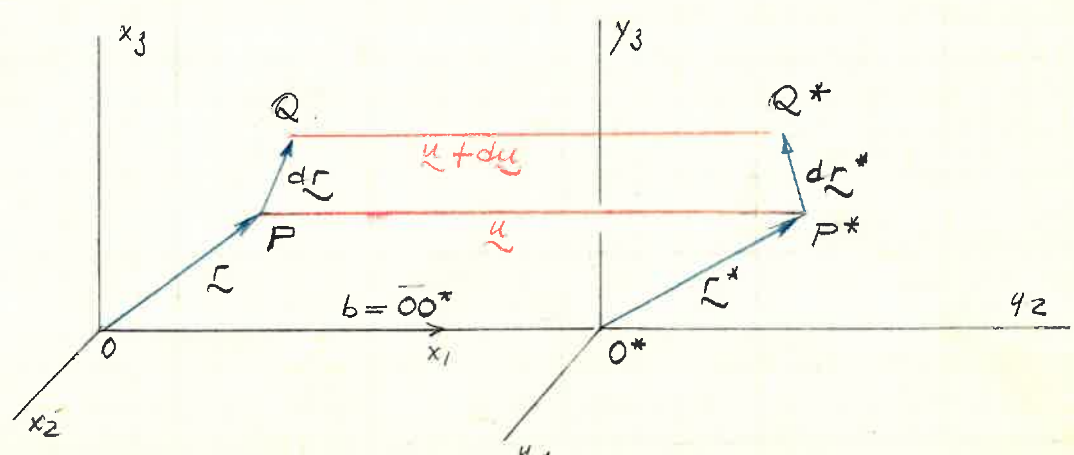
Special Strain Tensors

$$\begin{vmatrix} E & 0 & 0 \\ & E & 0 \\ & & E \end{vmatrix}$$ Hydrostatic tensor

$$\begin{vmatrix} E_{11} & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{vmatrix}$$ Uniaxial extension (lateral surfaces are not free)

$$\begin{vmatrix} E_{11} & 0 & 0 \\ & E_{22} & 0 \\ & & E_{33} \end{vmatrix}$$ Simple extension along one axis

DISPLACEMENTS. Relationship with strains



$$b + \underset{\sim}{r}^* = \underset{\sim}{r} + \underset{\sim}{u}$$

if $b = 0$ $\qquad \underset{\sim}{r}^* = \underset{\sim}{r} + \underset{\sim}{u}$

b is independent of x, y.

$$ds^2 = d\underset{\sim}{r} \cdot d\underset{\sim}{r}$$

$$ds^{*2} = d\underset{\sim}{r}^* \cdot d\underset{\sim}{r}^*$$

differentiating $b + \underset{\sim}{r}^* = \underset{\sim}{r} + \underset{\sim}{u}$ we get $d\underset{\sim}{r}^* = d\underset{\sim}{r} + d\underset{\sim}{u}$. Then

$$ds^{*2} = (d\underset{\sim}{r} + d\underset{\sim}{u})(d\underset{\sim}{r} + d\underset{\sim}{u}) = d\underset{\sim}{r} \cdot d\underset{\sim}{r} + d\underset{\sim}{u}\, d\underset{\sim}{u} + 2 d\underset{\sim}{r}\, d\underset{\sim}{u}$$

$$ds^{*2} - ds^2 = 2 d\underset{\sim}{r} \cdot d\underset{\sim}{u} + d\underset{\sim}{u} \cdot d\underset{\sim}{u}$$

but $\qquad d\underset{\sim}{u} = u_{j,k}\, dx_k\, \hat{i}_j$

$$d\underset{\sim}{u} \cdot d\underset{\sim}{u} = u_{j,k} \cdot u_{i,l} \cdot dx_k \cdot dx_l \cdot \hat{i}_j \cdot \hat{i}_i =$$

$$= u_{j,k}\, u_{i,l} \cdot dx_k \cdot dx_l\, \delta_{ij}$$

$$2 d\underset{\sim}{r}\, d\underset{\sim}{u} = 2 u_{j,k} \cdot dx_k\, \hat{i}_j \cdot dx_l \cdot \hat{i}_l = 2 u_{j,k}\, dx_k\, dx_l\, \delta_{jl}$$

as $\delta_{ij} \neq 0$ only if $i = j$ and same for δ_{jl} ($j = l$) we get

$$d\underset{\sim}{u} \cdot d\underset{\sim}{u} = u_{j,k}\, u_{j,l}\, dx_k\, dx_l = u_{k,i} u_{k,j}\, dx_i\, dx_j$$

$$2 d\underset{\sim}{r} \cdot du = 2 u_{j,k}\, dx_k\, dx_j = 2 u_{i,j}\, dx_i\, dx_j$$

$$ds^{*2} - ds^2 = (2 u_{i,j} + u_{k,i} u_{k,j})\, dx_i\, dx_j = 2 E_{ij}\, dx_i dx_j$$

now we decompose:

$$2 u_{i,j} = \underbrace{(u_{i,j} + u_{j,i})}_{\text{symmetric}} + \underbrace{(u_{i,j} - u_{j,i})}_{\text{antisymmetric}} = 2(\epsilon_{ij} + \omega_{ij})$$

and noting that $(u_{i,j} - u_{j,i})\, dx_i dx_j = 2\omega_{ij}\, dx_i\, dx_j = 0$

$$u_{i,j}\, dx_i dx_j = \epsilon_{ij}\, dx_i dx_j$$

Then

$$2 E_{ij} = (2 \epsilon_{ij} + u_{k,i} u_{k,j})\, dx_i\, dx_j$$

$$\boxed{E_{ij} = \epsilon_{ij} + \frac{1}{2} u_{k,i} u_{k,j}}$$

or $$\boxed{E_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) + \frac{1}{2} u_{k,i} u_{k,j}}$$

are the so-called <u>strain-displacement equations</u>

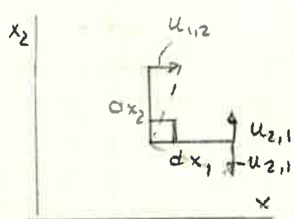
Using the decomp. above $u_{n,i} = \epsilon_{in} + w_{ni}$ $\quad u_{m,j} = \epsilon_{jm} + w_{mj}$
the last term may be written

$$u_{k,i} u_{k,j} = \frac{1}{2} \delta_{mn} (\epsilon_{in} + w_{ni})(\epsilon_{jm} + w_{mj})$$

and

$$E_{ij} = \epsilon_{ij} + \frac{1}{2} \delta_{mn} (\epsilon_{in} + w_{ni})(\epsilon_{jm} + w_{mj})$$

The quantities w_{ni} have the meaning of whole rotations of the body (as rigid) see figure



average rotation of $\perp$ dx_1, dx_2

$$w_{12} = \frac{1}{2}(u_{1,2} - u_{2,1})$$

Then the second order term $\frac{1}{2} u_{i,j} u_{j,i}$ consists of terms:

$$\epsilon^2, \quad \omega^2, \quad \epsilon\omega$$

if deformations are very small $\epsilon^2 << \epsilon$ $\quad \epsilon^2 \sim 0$
rotations " $\omega^2 << \epsilon$ $\quad \omega^2 \sim 0$

there are many cases in which it is necessary to take account of the rotations ω which are of the order or larger than ϵ's
In the case we neglect both second order terms in ϵ or ω we simply write

$$\boxed{E_{ij} = \epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})}$$

which includes <u>two</u> linearizations. Here $E_{ij} = \epsilon_{ij}$ is the "classical" linear strain tensor

---

If x_i, y_i are in the same coordinate system

$u_i = y_i - x_i$ $\qquad u, v, w$ classical notation

$x_i =$ $\qquad x, y, z$

$$E_{11} = E_{xx} = \frac{\partial u}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right]$$

$$E_{12} = E_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + \frac{1}{2}\left[\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right]$$

etc and the classical linear SD theory

$$E_{xx} = \epsilon_x = \frac{\partial u}{\partial x}$$

$$E_{xy} = \frac{1}{2}\gamma_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \quad \text{etc}$$

In many non-linear problems, as buckling of plates and shell, terms $u_{k,i}\, u_{k,j}$ must be included (at least the rotations ω's)

Linear strain tensor

$$\epsilon_{ij} = \begin{pmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ & \epsilon_{22} & \epsilon_{23} \\ & & \epsilon_{33} \end{pmatrix} \qquad \epsilon_{ij} = \epsilon_{ji}$$

in a x'_i coordinate system

$$x_i = a_{ij}\, x'_j$$

$$\epsilon_{ij}(x') = a_{im}\, a_{jn}\, \epsilon_{mn}$$

In the x, y, z coordinates, the classic notation is

$$\epsilon_{ij} = \begin{pmatrix} \epsilon_x & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ & \epsilon_y & \frac{1}{2}\gamma_{yz} \\ & & \epsilon_z \end{pmatrix}$$

The geometrical typical way of deriving the classic SD equations is given in many books on elasticity see f.i Timoshenko & Goodier.

$$\epsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$$

in the 3-dim case we have

6 components of strains at a point
3 " of displacements

Then: There are 3 conditions between the ϵ_{ij}. These expressions arise from the condition of compatibility of deformation.

Compatibility equations

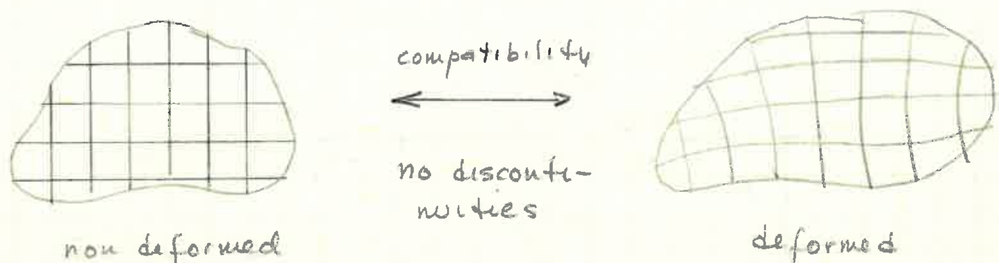
In two dimensions

$$\epsilon_{xx} = u_{,x} \quad \leftarrow (1)$$
$$\epsilon_{yy} = v_{,y} \quad \leftarrow (2)$$
$$\epsilon_{xy} = \tfrac{1}{2}(u_{,y} + v_{,x}) \quad \leftarrow (3)$$

$$u(x,y) = \int \epsilon_{xx}\,dx + f(y)$$

$$v(x,y) = \int \epsilon_{yy}\,dy + g(x)$$

and (3) must be satisfied. This has the physical meaning of



The compatibility eq. in 2 dim ~~is easily~~ can be readily obtained ~~in~~ this way:

$$\epsilon_{xx,yy} = u_{,xyy} \qquad \text{from (1)}$$

$$\epsilon_{yy,xx} = v_{,yxx} \qquad \text{from (2)}$$

$$2\,\epsilon_{xy,xy} = u_{,yxy} + v_{,xyx} \qquad \text{from (3)}$$

Then

$$\boxed{\epsilon_{xx,yy} + \epsilon_{yy,xx} = 2\,\epsilon_{xy,xy}}$$

In the 3 dim. case (see for instance Sokolnikoff)

$$\boxed{\epsilon_{ij,kl} + \epsilon_{kl,ij} = \epsilon_{ik,jl} + \epsilon_{jl,ik}}$$

and there are only "6" independent equations (rigurously only 3). To see the classical notation see Timoshenko, chapter 8.

Notes: 1) we have always referred the equations to the initial coordinate system (non-deformed body)

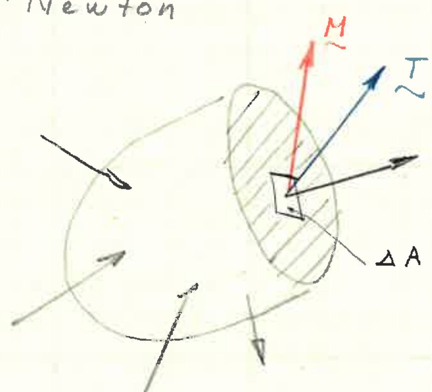
2) if the non-linear terms are to be included comp. equations are of course different and more complex.

FORCES ACTING IN A BODY

External effects on a body

- Body forces
  1. Body forces per unit mass (ex: weight, fuerza centrifuga)
  2. Body couples per unit mass (ex: magnetism) are seldom considered
- Contact forces
  3. Surface contact forces per unit of area → surface stress vector
  4. Surface contact couple per unit of area → surface couple stress vector

Internal mutual action between pairs of particles. Internal force vectors must vanish from 3rd law of Newton



$\underset{\sim}{M}$, $\underset{\sim}{T}$ are resultant couple & force vector on plane $\hat{n}$.

(Plane $\hat{n}$ defined by its normal $\hat{n}$.)

Stress vector for plane $\hat{n}$ $\qquad \underset{\sim}{t}^{n} = \lim_{\Delta A \to 0} \dfrac{\Delta \underset{\sim}{T}}{\Delta A}$

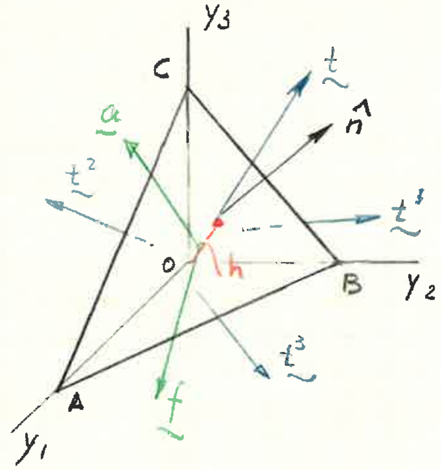
positive if the angle with $\hat{n}$ is $< 90°$

Couple stress for plane $\hat{n}$ $\qquad \underset{\sim}{m}^{n} = \lim_{\Delta A \to 0} \dfrac{\Delta \underset{\sim}{M}}{\Delta A}$

it is optative to include $\underset{\sim}{m}$ in the Theory. It is generally assumed $\underset{\sim}{m} \cong 0$. (higher order than $\underset{\sim}{t}$)

We relate now the stress vector a to R.C.C. system

(see figure next page)



y_1, y_2, y_3 coordinates in the deformed state

n_1, n_2, n_3 direction cosines of $\hat{n}$

t_1, t_2, t_3 components of $\underset{\sim}{t}$

$\underset{\sim}{t}^1, \underset{\sim}{t}^2, \underset{\sim}{t}^3$ stress vectors for planes $\perp$ y_1, y_2, y_3

$\underset{\sim}{f}$ body force per unit of volume components f_1, f_2, f_3

$\underset{\sim}{a}$ acceleration force (per unit of volume) a_1, a_2, a_3

Area ABC $= \Delta A$
BOC $= \Delta A \, n_1$
AOC $= \Delta A \, n_2$
AOB $= \Delta A \, n_3$

The sum of forces in the y_i direction must be zero:

$$t_i \Delta A - t_i^1 \Delta A \, n_1 - t_i^2 \Delta A \, n_2 - t_i^3 \Delta A \, n_3 + \rho(\Delta V) f_i = \rho(\Delta V) a_i$$

as $\Delta V = \frac{h}{3} \Delta V$ when $h \to 0$ $\Delta V / \Delta A \to 0$ (3rd order). Cancelling ΔA we get

$$t_i - (t_i^1 n_1 + t_i^2 n_2 + t_i^3 n_3) = 0$$

$$\boxed{t_i = t_i^j n_j} \qquad i, j = 1, 2, 3$$

We define $\boxed{t_i^j = \tau_{ij}}$ $\to$ stress tensor or Cauchy tensor or rank 2

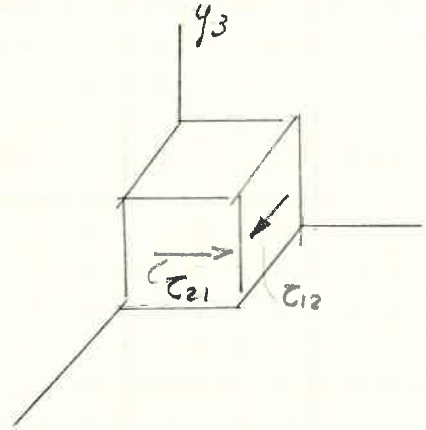
as the 9 components of the stress vector acting in CS.

$$\boxed{t_i = \tau_{ij} n_j} \qquad \tau_{ij} = n_{ij} a_j$$

$$\tau_{ij} = \begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{pmatrix}$$

classical notation (Timoshenko)

In a y' coordinate system $\tau'_{ij} = a_{mi} a_{nj} \tau_{mn}$ in two dimension corresponds to Mohr transf.

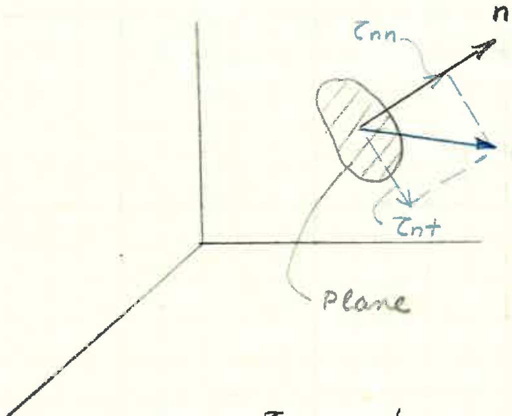


If no body couples are present it is very easy to show that

$$\boxed{\tau_{ij} = \tau_{ji}}$$

taking moment about the normal axes and observing that the contribution of the body force is of 4th order

Normal and Shearing stress



Normal stress $\tau_{nn} = \hat{n} \cdot \underset{\sim}{t}$

Shearing stress τ_{nt}

$$\tau_{nn} = t_i n_i = \tau_{ji} n_j n_i = \tau_{11} n_1^2 + \tau_{22} n_2^2 +$$
$$+ \tau_{33} n_3^2 + 2\tau_{12} n_1 n_2 + 2\tau_{23} n_2 n_3 + 2\tau_{31} n_3 n_1$$

n_1, n_2, n_3 are often called l, m, n (Timoshenko)

If x_1, x_2, x_3 are the principal directions

$$\tau_{nn} = \tau_1 n_1^2 + \tau_2 n_2^2 + \tau_3 n_3^2$$

The shear stress τ_{nt} is

$$\tau_{nt} = \sqrt{|t_i|^2 - \tau_{nn}^2}$$

Principal directions and stresses

τ_{ij} given, principal direction and stresses are obtained the same way we described for the strain tensor. The cubic equation is here

$$\tau^3 - \Theta_1 \tau^2 + \Theta_2 \tau - \Theta_3 = 0$$

the invariants being

$$\Theta_1 = \tau_{11} + \tau_{22} + \tau_{33} = \tau_1 + \tau_2 + \tau_3$$

$$\Theta_2 = \tau_{11}\tau_{22} + \tau_{22}\tau_{33} + \tau_{33}\tau_{11} - \tau_{12}^2 - \tau_{23}^2 - \tau_{31}^2 =$$

$$= \tau_1\tau_2 + \tau_2\tau_3 + \tau_3\tau_3$$

$$\Theta_3 = \tau_{11}\tau_{22}\tau_{33} + \; \cdots\cdots \; = \tau_1\tau_2\tau_3$$

Got the roots $\tau_1 > \tau_2 > \tau_3$, substitute into $(\tau_{ij} - \tau\,\delta_{ij})\,n_j = 0$ to find n_j's

<u>Resolution into a volumetric and a deviatoric stress tensor</u>

We define a deviatoric stress tensor S_{ij} as

$$\boxed{\tau_{ij} = S_{ij} + \frac{1}{3}\delta_{ij}\,\tau_{KK}}$$

or $\quad \tau_{ij} = S_{ij} + \frac{1}{3}\delta_{ij}\,\Theta_1$

$\frac{1}{3}\tau_{KK}$ or $\frac{1}{3}\Theta_1$ is the so-called <u>mean stress</u> at a point, σ_m

Then

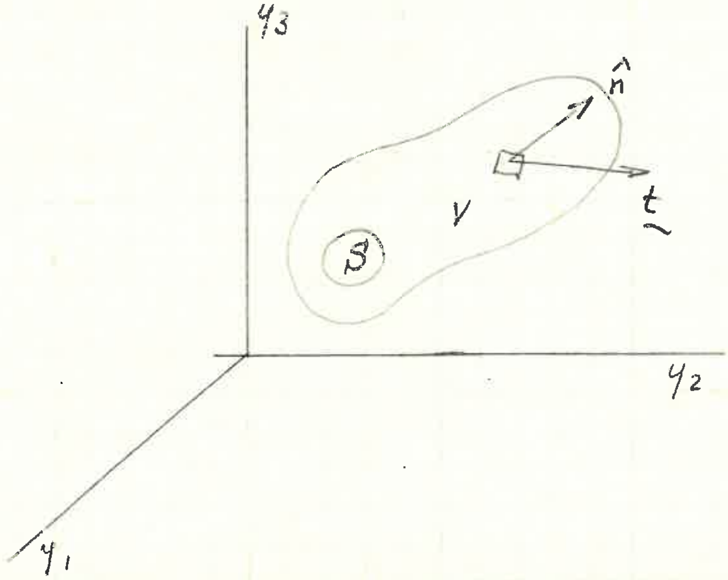
$$\tau_{ij} = \underset{(1)}{\begin{pmatrix} \sigma_m & 0 & 0 \\ & \sigma_m & 0 \\ & & \sigma_m \end{pmatrix}} + \underset{(2)}{\begin{pmatrix} \tau_{11}-\sigma_m & \tau_{12} & \tau_{13} \\ & \tau_{22}-\sigma_m & \tau_{23} \\ & & \tau_{33}-\sigma_m \end{pmatrix}}$$

(1) $\rightarrow$ volumetric, hydrostatic or isotropic stress tensor (diagonal tensor)

(2) $\rightarrow$ deviatoric stress tensor.

The reasons for this resolution will be clear when speaking of plastic behavior and stress energy.

Stress equations of equilibrium



Consider the body in its deformed state

S surface

V volume

Equilibrium along any axis i requires

$$\int_S t_i \, dS + \int_V \rho f_i \, dV = \int_V \rho a_i \, dV$$

(a_i: component of acceleration vector) Substitute now $t_i = \tau_{ij} n_j$

$$\int_S \tau_{ij} n_j \, dS + \int_V \rho f_i \, dV = \int_V \rho a_i \, dV$$

to reduce the 1st integral to a volume integral we use the Gauss' divergence theorem

$$\int_S \underset{\sim}{A} \, \hat{n} \, dS = \int_V \operatorname{div} \underset{\sim}{A} \, dV$$

then

$$\int_V [\tau_{ij,j} + \rho f_i - \rho a_i] \, dV = 0$$

for arbitrary volume, hence

$$\boxed{\tau_{ij,j} + \rho f_i - \rho a_i = 0} \qquad i = 1, 2, 3$$

The stress equilibrium equations of Cauchy. In R.C.C with Timoshenko's notation

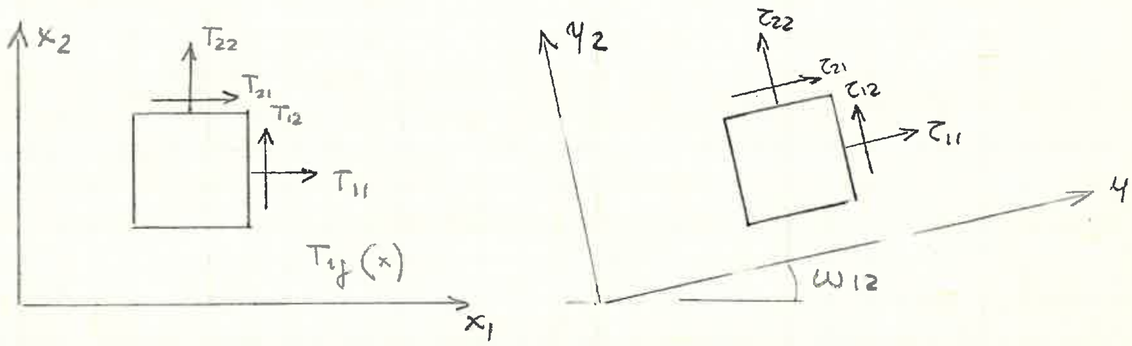
$$\rho f_i = (X, Y, Z)$$

$i = 1 = x \qquad \dfrac{\partial \sigma_x}{\partial x} + \dfrac{\partial \tau_{xy}}{\partial y} + \dfrac{\partial \tau_{xz}}{\partial z} + X = \rho a_x$

etc. Note that the differentiation is taken with respect to the <u>deformed</u> state

Expression of Equilibrium Equations for the non deformed body, using small deformation theory

We include only effect of the rotation of a volume element



$x \to y$ coordinate transformation is a rigid body tranformation for which

$$ds^{*2} - ds^2 = 0$$

$$\boxed{\tau_{ij} = a_{mi}\, a_{nj}\, T_{mn}}$$

T_{mn} Kirchoff tensor, a_{mi}, a_{nj} are function of x_i or y_j.

This is not a cartesian transformation, for which a_{mi} a_{nj} are constant

$$\frac{\partial y_j}{\partial x_i} = a_{ij}$$

The displacement vectors u_j verify

$$y_j = x_j + u_j \quad \therefore \quad \frac{\partial y_j}{\partial x_i} = \delta_{ij} + u_{j,i}$$

but
$$u_{j,i} = \overset{\text{assume } 0}{\cancel{\epsilon_{ij}}} + \omega_{ji} = \omega_{ji}$$

$$\frac{\partial y_j}{\partial x_i} = \delta_{ij} + \omega_{ji} = a_{ij}$$

and substituting:

$$\tau_{ij} = (\delta_{im} + \omega_{mi})(\delta_{jn} + \omega_{nj})\, T_{mn}$$

we shall neglect $\omega_{ij}^2 << 1$ (not ω_{ij}) then

$$\boxed{\tau_{ij} = T_{ij} + \omega_{mi} T_{mj} + \omega_{mj} T_{mi}}$$

Now
$$\frac{\partial \tau_{ij}}{\partial y_j} = \overset{\to 0}{\cancel{\frac{\partial^2 y_j}{\partial y_j \partial x_m}}}\left[\frac{\partial y_i}{\partial x_m} T_{mn}\right] + \frac{\partial y_j}{\partial x_n}\frac{\partial}{\partial y_j}\left(\frac{\partial y_i}{\partial x_m} T_{mn}\right)$$

$$= \frac{\partial y_j}{\partial x_n}\frac{\partial}{\partial x_k}\left(\frac{\partial y_i}{\partial x_m} T_{mn}\right)\frac{\partial x_k}{\partial y_j} = \delta_{kn}\frac{\partial}{\partial x_k}\left(\frac{\partial y_i}{\partial x_m} T_{mn}\right) =$$

$$= \frac{\partial}{\partial x_n}\left(\frac{\partial y_i}{\partial x_m} T_{mn}\right) = \frac{\partial}{\partial x_n}\left[\left(\delta_{mi} + \omega_{mi}\right) T_{mn}\right]$$

and the equilibrium equations are

$$\boxed{\frac{\partial}{\partial x_n}\left[\left(\delta_{mi} + \omega_{im}\right) T_{mn}\right] + \rho f_i = \rho a_i}$$

If f_i and $a_i \sim 0$ (neither body nor inertia forces):

$$\boxed{\frac{\partial}{\partial x_n}\left[\left(\delta_{mi} + \omega_{im}\right) T_{mn}\right] = 0}$$

Example: for $i = 1$, $m, n = 1, 2, 3$ The equation is

$$\frac{\partial}{\partial x_1}\left[\left(\delta_{11} T_{1n} + \omega_{12} T_{2n} + \omega_{13} T_{3n}\right] = 0$$

or
$$\frac{\partial T_{11}}{\partial x_1} + \frac{\partial T_{12}}{\partial x_2} + \frac{\partial T_{13}}{\partial x_3} + \frac{\partial}{\partial x_1}\left(\omega_{12} T_{21}\right) + \frac{\partial}{\partial x_2}\left(\omega_{12} T_{22}\right) +$$

$$+ \frac{\partial}{\partial x_3}\left(\omega_{12} T_{23}\right) + \frac{\partial}{\partial x_1}\left(\omega_{13} T_{31}\right) + \frac{\partial}{\partial x_2}\left(\omega_{13} T_{32}\right) + \frac{\partial}{\partial x_3}\left(\omega_{13} T_{33}\right) = 0$$

If $\omega \approx 0$ $\tau_{ij} = T_{ij}$ we are reduced to the first three terms (classical linear theory):

$$\frac{\partial}{\partial x_n}\left[\delta_{im} T_{mn}\right] = T_{in,n} = 0$$

or $\boxed{\tau_{ij,j} = 0}$

REFERENCE Novozhilov "Non linear theory of elasticity"

CONSTITUTIVE EQUATIONS

To deal with this problem, we have to introduce and define ideal classes of materials, either by definition or as a postulate.

Postulates are introduced on basis of experimental knowledge. But "experimental knowledge is incapable of leading toward a general Theory".

Guide Lines for Formulating a Theory

Some of the more important principles, as stated by Truesdell in his book, are the following ones:

1. **Exclusion principles.** neither the proper set of independent variables nor the function form related can be established a priori. At most an exclussion of certain variables can be effected. Some consequences:

1a) Principle of Heredity (Volterra) the mechanical behavior of materials at a time t is specified in terms of the past behavior of the body up to time t

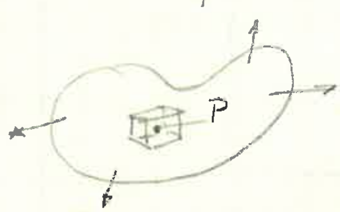
Examples:

stress $\tau = f(\text{strain } \epsilon) \rightarrow$ elastic material

$\tau = f[\epsilon(t)] \rightarrow$ viscid material time dependent

time $t' > t$ has no influence and is excluded

1b) Principle of Neighborhood: The mechanical behavior of a material at a point x occupying point $y(x,t)$ is specified in terms of an arbitrary small neighborhood of x



$$ds^{*2} - ds^2 = 2E_{ij}$$

This principle should be carefully applied if we included electromagnetic effects

1c) Principle of Equipresence: a variable present in one constitutive equation for a material should be present in all constitutive equations. All equations should content the same independent variables

Example: Thermoelastic effect must include

$[\tau_{ij}, \epsilon_{ij}, T]$ f. instance $\epsilon_x = \frac{\sigma_x}{E} + \alpha T$

Thermodynamic effect must include

$[\tau_{ij}, \epsilon_{ij}, T, t]$ such as the Fourier equation

for heat flow $\nabla^2 T = k \frac{\partial T}{\partial t} + e \frac{\partial}{\partial t}(\epsilon_{kl})$

2. **Coordinate Invariance**: variables must be connected through equations that are invariants under any coordinate transformation.
(This is automatically fulfilled by using the tensor notation)

3. **Spatial Invariance**: constitutive equations must be invariants to rigid motion of the coordinates in the deformed body

4. **Material Invariance**: constitutive equations which are invariant under a group of transformation of the coordinates are said to have a symmetry characterized by the group

Ex; isotropic materials: invariant under all groups
ortotropic " : " " reflexions, etc

5. **Dimensional Invariance**: constitutive equations must be formulated such that Buckingham's Theorem holds true.

References for Material Science

Nadai Theory of Fracture & Flow (A little old-fashioned now)
Freudenthal Inelastic Behavior of Engineering Materials.

ELASTIC SOLIDS

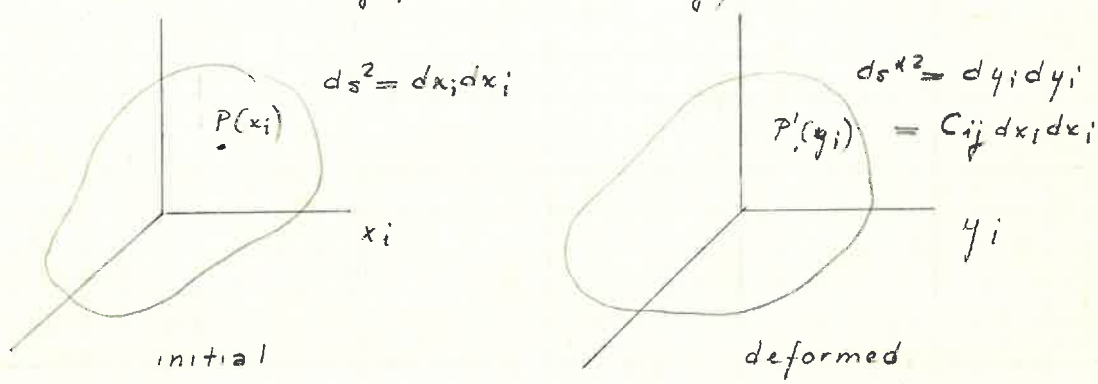
An ES is characterized by two configurations:

1) natural state or nondeformed state
2) deformed state, attainable by any reversible process.

This is associated with the idea of potential, strain or stress energy. The behavior of an ES is independent from the precedent history of the material.

Let's define $U \Longrightarrow$ strain energy per unit of deformed volume (strain energy density)

$$U = U(x_i, E_{ij}) \text{ or } U(x_i, C_{ij})$$



$$\underset{\substack{\uparrow \\ \text{deformation}}}{C_{ij}} = 2 \underset{\substack{\uparrow \\ \text{strain}}}{E_{ij}} + \delta_{ij}$$

Properties:

a) U is a scalar invariant, unaffected by rigid body displacements

b) If U is independent of x_i the material is said to be homogeneous

For an isotropic homogeneous material, taking the principal strain axes,

$$E_{ij} = \begin{bmatrix} E_1 & & \\ 0 & E_2 & \\ 0 & 0 & E_3 \end{bmatrix}$$

$$U = U(E_1, E_2, E_3)$$

and since strain-stress law is independent of directions in the material

$$U = U(\theta_1, \theta_2, \theta_3) \qquad \theta_i = \text{invariants of the strain tensor.}$$

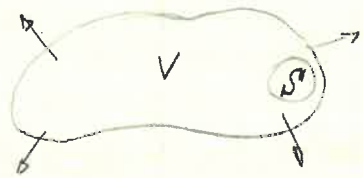
Principle of Virtual Work

Virtual work done by all the forces acting on a body in equilibrium during any virtual displacement $\delta \underset{\sim}{u}$ (infinitesimal change in geometric configuration) must be zero

$$\delta W = 0$$

$$W = W_e + W_i$$

W_e work done by external forces
W_i " " internal "

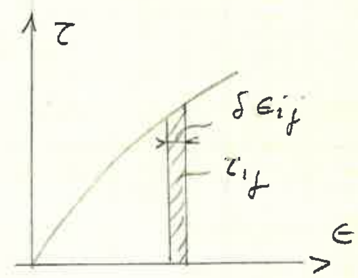


$$\int \tau_{ij}\, \delta u_i\, dS + \int \rho f_i\, \delta u_i\, dV + W_i = 0$$

$$+ \int_V \delta (dW_i)$$

$$- \int \delta U\, dV$$

For small deformation theory



$$U = U(\epsilon_{ij})$$

$$\delta U = \frac{\partial U}{\partial \epsilon_{ij}} \delta \epsilon_{ij} = \frac{\partial U}{\partial \epsilon_{11}} \delta \epsilon_{11} + \frac{\partial U}{\partial \epsilon_{22}} \delta \epsilon_{22} + \cdots\cdots$$

$$\left(\tau_{ij} - \frac{\partial U}{\partial \epsilon_{ij}}\right) \delta \epsilon_{ij} = 0 \quad \therefore \Rightarrow \quad \tau_{ij} = \frac{\partial U}{\partial \epsilon_{ij}}$$

If we are not dealing with small deformation, but with a large one, we have to look for a

Form of Strain Energy Function (Green, 1839)

In an isotropic homogeneous body U is independent of direction

$$U = U(\epsilon_1, \epsilon_2, \epsilon_3) \qquad \epsilon_i = \text{principal strains}$$

or alternatively we may use invariants of the strain tensor as arguments of U:

$$U = U(\theta_1, \theta_2, \theta_3)$$

$$\theta_1 = \epsilon_1 + \epsilon_2 + \epsilon_3 \qquad \text{order } \epsilon$$

$$\theta_2 = \epsilon_1\epsilon_2 + \epsilon_2\epsilon_3 + \epsilon_3\epsilon_1 \qquad \text{order } \epsilon^2$$

$$\theta_3 = \epsilon_1\epsilon_2\epsilon_3 \qquad \text{order } \epsilon^3$$

θ_1, θ_2 and θ_3 form a basis because any other invariant can be expressed in terms of them

$$\tau_{ij} = \frac{\partial U}{\partial \theta_1}\frac{\partial \theta_1}{\partial \epsilon_{ij}} + \frac{\partial U}{\partial \theta_2}\frac{\partial \theta_2}{\partial \epsilon_{ij}} + \frac{\partial U}{\partial \theta_3}\frac{\partial \theta_3}{\partial \epsilon_{ij}}$$

One possible way of expressing U is to expand it in power series:

$$U = \sum_{i,j,k} A_{ijk}\, \theta_1^{\,i}\, \theta_2^{\,j}\, \theta_3^{\,k}$$

or

$$U = \underbrace{A_{000} + A_{100}\theta_1 + A_{200}\theta_1^2 + A_{010}\theta_2} + \underbrace{\cdots\cdots}_{\text{order } \epsilon^3, \epsilon^4 \ldots}$$

Underlined terms define a linear elastic solid; strain energy U is a quadratic function of ϵ's

$$\tau_{ij} = A_{100}\frac{\partial \theta_1}{\partial \epsilon_{ij}} + 2A_{200}\theta_1\frac{\partial \theta_1}{\partial \epsilon_{ij}} + A_{010}\frac{\partial \theta_2}{\partial \epsilon_{ij}} + \cdots \to 0$$

$$\theta_1 = \epsilon_{kk} \qquad \frac{\partial \theta_1}{\partial \epsilon_{ij}} = \delta_{ij}$$

$$\theta_2 = \frac{1}{2}\left[\theta_1^2 - \delta_{mn}\delta_{kl}\,\epsilon_{mk}\,\epsilon_{nl}\right] \qquad \frac{\partial \theta_2}{\partial \epsilon_{ij}} = \theta_1\delta_{ij} - \epsilon_{ij}$$

$$\tau_{ij} = A_{100}\delta_{ij} + 2A_{200}\theta_1\delta_{ij} + A_{010}(\theta_1\delta_{ij} - \epsilon_{ij})$$

we may chose $A_{000} = 0$ (no influence on stresses) and we must chose $A_{100} = 0$ so that $\tau_{ij} = 0$ when $\epsilon_{ij} = 0$ (no residual stresses assumed)

Then we get

$$\tau_{ij} = (A_{010} + 2A_{200})\theta_1\delta_{ij} - A_{010}\epsilon_{ij}$$

Let's call

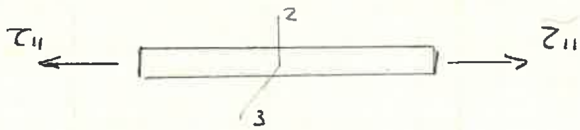
$$\left.\begin{aligned} A_{010} + 2A_{200} &= \lambda \\ -A_{010} &= 2\mu \end{aligned}\right\} \text{LAMÉ constants}$$

$$\boxed{\tau_{ij} = \lambda \theta_1 \delta_{ij} + 2\mu \epsilon_{ij}}$$ Generalized Hooke's law

constitutive equations for elastic solids

The physical meaning of this parameters can be easely found relating them to some usual tests:

Uniaxial Test $\tau_{11} \neq 0$ others zero



and for an isotropic material $\epsilon_{22} = \epsilon_{33}$ $\epsilon_{11} \neq 0$.

$$\tau_{11} = \lambda \theta_1 + 2\mu \epsilon_{11} = \lambda(\epsilon_{11} + 2\epsilon_{22}) + 2\mu \epsilon_{11}$$

$$0 = \lambda \theta_1 + 2\mu \epsilon_{22} = \lambda(\epsilon_{11} + 2\epsilon_{22}) + 2\mu \epsilon_{22}$$

whence

$$\frac{\tau_{11}}{\epsilon_{11}} = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu} = E$$ modulus of elasticity extensional modulus

$$\frac{\epsilon_{22}}{\epsilon_{11}} = -\frac{\lambda}{2(\lambda+\mu)} = -\nu$$ Poisson's ratio.

Pure shear test $\epsilon_{11} = \epsilon_{22} = \epsilon_{33} = 0$

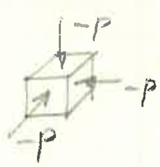
$$2\epsilon_{ij} = \gamma_{ij}$$

$$\frac{\tau_{ij}}{\gamma_{ij}} = \mu = G$$ shear modulus.

Triaxial pressure $\tau_{11} = \tau_{22} = \tau_{33} = -p$ others zero

Let $i = j$ in the stress-strain law

$$\tau_{ii} = \Theta_1 = 3\lambda \theta_1 + 2\mu \epsilon_{ii} = (3\lambda + 2\mu)\theta_1$$



$$\Theta_1 = -3p \qquad \theta_1 = \epsilon_{11} + \epsilon_{22} + \epsilon_{33} = e$$

$$-3p = (3\lambda + 2\mu)e$$

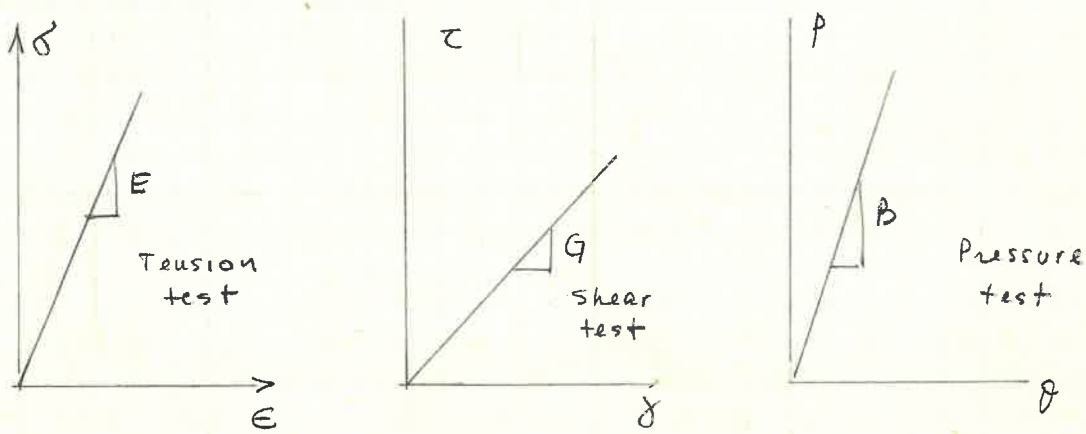
The Bulk modulus B is

$$B = \frac{\text{Pressure}}{\Delta V / V} = \frac{p}{\theta_1} = \frac{3\lambda + 2\mu}{3} = \lambda + \frac{2}{3}\mu$$

The Lamé constants can also be expressed in terms of E, ν, G etc

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \qquad \mu = G = \frac{E}{2(1+\nu)}$$

$$B = \frac{E}{3(1-2\nu)}$$



$B > E > G.$

Strain-Stress law

$$\epsilon_{ij} = \frac{\tau_{ij}}{2\mu} - \frac{\lambda \theta_1 \delta_{ij}}{2\mu} = \frac{\theta_1}{3\lambda + 2\mu}$$

$$\boxed{\epsilon_{ij} = \frac{1+\nu}{E}\tau_{ij} - \frac{\nu}{E}\Theta_1 \delta_{ij}}$$

Strain Energy

$$U = \left(\frac{\lambda + 2\mu}{2}\right)\theta_1^2 - 2\mu\theta_2$$

$$= \frac{1}{2}\lambda(\epsilon_{kk})^2 + \mu\,\epsilon_{ij}\epsilon_{ij}$$

Complementary Strain Energy

$W = W(\Theta_1, \Theta_2)$ for a elastic linear isotropic material

$$W = \frac{1+\nu}{E}\tau_{ij}\tau_{ij} - \frac{\nu}{2E}\Theta_1^2$$

Properties: $\epsilon_{ij} = \frac{\partial W}{\partial \tau_{ij}}$ conjugate of $\tau_{ij} = \frac{\partial U}{\partial \epsilon_{ij}}$

ANISOTROPIC ELASTIC SOLIDS

Assumptions: small strains, small rotations. In this case we need the magnitudes of the invariants and moreover the direction of the principal stresses.

$$U = \frac{1}{2} C_{ijkl}\, \epsilon_{ij} \epsilon_{kl} \qquad \text{quadratic form}$$

$$C_{ijkl} = \text{Elastic Modulus tensor}$$

Some properties

$$\underbrace{C_{ijkl} = C_{jikl} \qquad C_{ijkl} = C_{ijlk}}_{\text{From symmetry of } \tau_{ij},\ \epsilon_{ij}} \qquad \underbrace{C_{ijkl} = C_{klij}}_{\text{From } U \geq 0 \text{ for arbitrary strain}}$$

These relation reduce the number of the different components from 81 to 21

$$C_{ijkl} \quad \begin{vmatrix} C_{1111} & C_{1122} & C_{1133} & - & - & - \\ C_{2211} & C_{2222} & C_{2233} & - & - & - \\ C_{3311} & C_{3322} & C_{3333} & - & - & - \\ - & - & - & C_{2323} & - & - \\ - & - & - & - & C_{1313} & - \\ - & - & - & - & - & C_{1212} \end{vmatrix}$$

sym

<u>Symmetries</u> A material is symmetric with respect to a particular coordinate transformation if the elastic modulus tensor is invariant under that particular transformation.

$$C'_{ijkl}(x) = a_{mi}\, a_{nj}\, a_{pk}\, a_{ql}\, C_{mnpq}(x)$$

Many materials have at least one plane of symmetry. Taking it as the $x_1 x_2$ plane

any transformation
$$x_1' = x_1$$
$$x_2' = x_2$$
$$-x_3 = x_3$$

leaves the tensor invariant. These condition reduce the 21 independent constants to $\boxed{13}$

Orthotropic Solids are materials with two planes of symmetry (ex: wood) The number of constants reduce to 9

$$\begin{vmatrix} C_{1111} & C_{1122} & C_{1133} & 0 & 0 & 0 \\ C_{2211} & C_{2222} & C_{2233} & 0 & 0 & 0 \\ C_{3311} & C_{3322} & C_{3333} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{2323} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{1313} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{1212} \end{vmatrix}$$

they are 3 elastic moduli C_{1111}, C_{2222} & C_{3333}
3 shear " $C_{2323}, C_{1313}, C_{1212}$
3 Poisson's r. " $C_{1122} = C_{2211}$, $C_{1133} = C_{3311}$
$C_{2233} = C_{3322}$.

differentiating U we get for an orthotropic material

$$\tau_{11} = C_{1111}\epsilon_{11} + C_{1122}\epsilon_{22} + C_{1133}\epsilon_{33} + C_{1122}\epsilon_{12} + C_{1133}\epsilon_{13} + C_{2233}\epsilon_{23}$$

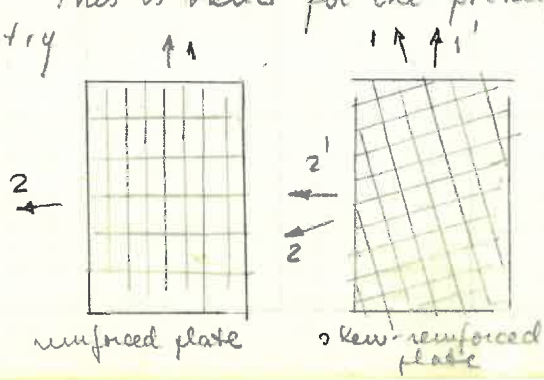
and similars for other stresses using $\tau_{ij} = C_{ijkl}\epsilon_{kl}$

For plane stress $\tau_{i3} = 0$

$$\tau_{11} = C_{1111}\epsilon_{11} + C_{1122}\epsilon_{22}$$
$$\tau_{22} = C_{2211}\epsilon_{22} + C_{2222}\epsilon_{22}$$
$$\tau_{12} = C_{1212}\epsilon_{12}$$

and we deal only with 4 constants C_{1111}, C_{2222}, $C_{1122} = C_{2211}$ & C_{1212}.

This is valid for the principal axes along the direction of symmetry



To refer to 1', 2' axes use tensor transform.

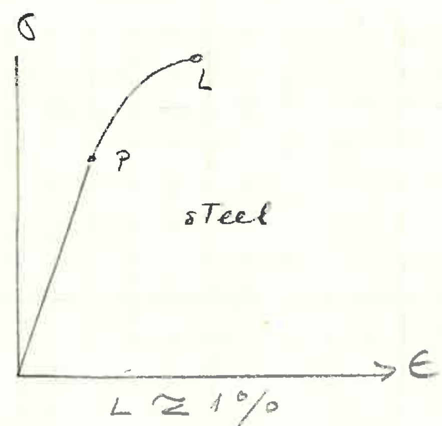
(see exercise 4, set 4)

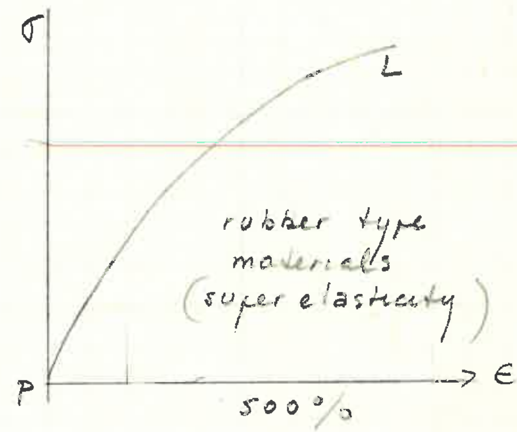
Ref: Love, Sokolnikoff.

NON-LINEAR ELASTICITY

NLE means: non-linear σ-ϵ curve but complete (or practically complete) reversible deformation

Two usual types of behavior:





P = proportional limit

PL = range of NLE

L = limit of elasticity (irrecoverable deformations)

This theory is very useful to be applied to new rubber type of materials, plastic, etc which are being developed. Two problems arise:

1) ~~the~~ non linear form of the curve

2) deformation are very large and can not be considered infinitesimal; it is more convenient to deal with the <u>deformation tensor</u> C_{mn} rather than with strain tensors.

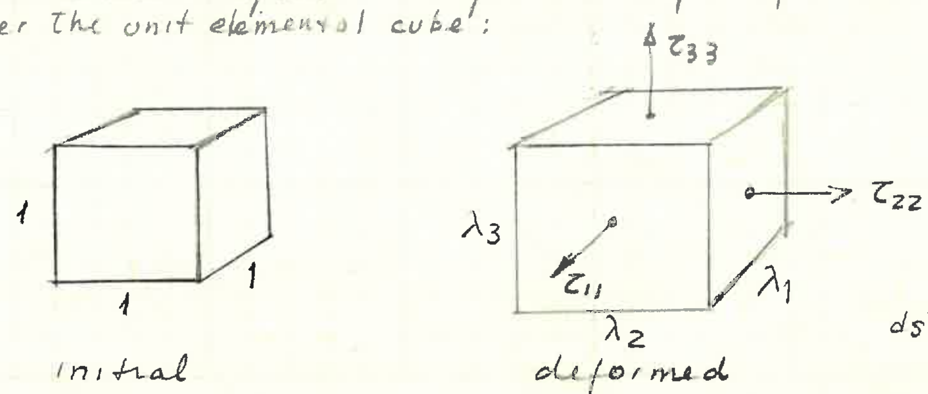
$$ds^{*2} = C_{ij}\, dx_i\, dx_j$$

$$C_{ij} = y_{k,i}\, y_{k,j} = 2E_{ij} + \delta_{ij}$$

Homogeneous state of deformation

We shall refer always to the principal axes

Consider the unit elemental cube:



$$ds^{*2} = C_{ij}\, dx_i\, dx_j$$

$$C_{ij} = \begin{pmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{pmatrix}$$

the invariants being

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$$
$$I_2 = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2$$
$$I_3 = \lambda_1^2\lambda_2^2\lambda_3^2$$

The strain energy U may be expressed as

$$U = U(I_1, I_2, I_3)$$

We can obtain the constitutive equations using the principle of virtual work

$$\delta(W_e - U) = 0$$

$$\delta W_e = \tau_{11}(\lambda_2\lambda_3)\,\delta\lambda_1 + \tau_{22}(\lambda_1\lambda_3)\,\delta\lambda_2 + \tau_{33}(\lambda_1\lambda_2)\,\delta\lambda_3$$

$$\delta U = \frac{\partial U}{\partial \lambda_1}\delta\lambda_1 + \frac{\partial U}{\partial \lambda_2}\delta\lambda_2 + \frac{\partial U}{\partial \lambda_3}\delta\lambda_3$$

since $\delta\lambda_i$ are independent;

$$(\lambda_2\lambda_3)\tau_{11} = \frac{\partial U}{\partial \lambda_1} \qquad (\lambda_3\lambda_1)\tau_{22} = \frac{\partial U}{\partial \lambda_2} \qquad (\lambda_1\lambda_2)\tau_{33} = \frac{\partial U}{\partial \lambda_3}$$

or $$(\lambda_2\lambda_3)\tau_{11} = \frac{\partial U}{\partial I_1}\frac{\partial I_1}{\partial \lambda_1} + \frac{\partial U}{\partial I_2}\frac{\partial I_2}{\partial \lambda_1} + \frac{\partial U}{\partial I_3}\frac{\partial I_3}{\partial \lambda_1} \qquad \text{etc.}$$

substituting

$$\frac{\partial I_1}{\partial \lambda_1} = 2\lambda_1 \qquad \frac{\partial I_2}{\partial \lambda_1} = 2\lambda_1(\lambda_2^2 + \lambda_3^2) \qquad \frac{\partial I_3}{\partial \lambda_1} = 2\lambda_1\lambda_2^2\lambda_3^2$$

we get

$$\tau_{11} = \frac{2\lambda_1}{\lambda_2\lambda_3}\left[\frac{\partial U}{\partial I_1} + (\lambda_2^2 + \lambda_3^2)\frac{\partial U}{\partial I_2} + \lambda_2^2\lambda_3^2\frac{\partial U}{\partial I_3}\right]$$

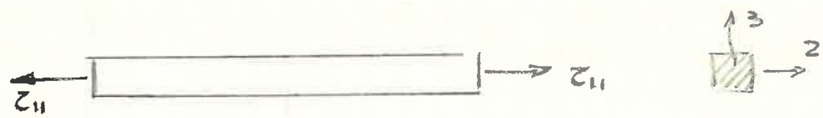
---

and similar ones for τ_{22} and τ_{33}

The expression of U is usually written as a series of powers of the I_i. Because of the meaning of U, an expansion in terms of $(I_i - I_i(1))$ is suggested (to have $U = 0$ when $\lambda = 1$, $\epsilon = 0$)

$$U = \sum_{i,j,k=1}^{\infty} C_{ijk} (I_1 - 3)^i (I_2 - 3)^j (I_3 - 1)^k$$

Simple tension test

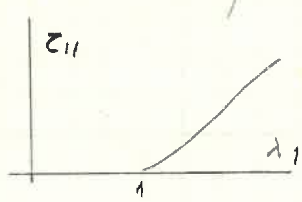


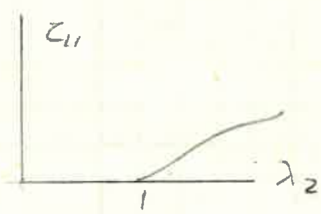
$\tau_{11} \neq 0$ others zero $\quad \epsilon_{22} = \epsilon_{33}$ (isotropic) $\therefore \lambda_2 = \lambda_3$

$$\tau_{11} = 2\lambda_1 \left[\frac{1}{\lambda_2} \frac{\partial U}{\partial I_1} + 2 \frac{\partial U}{\partial I_2} + \lambda_2^2 \frac{\partial U}{\partial I_3} \right]$$

$$0 = \frac{1}{\lambda_1 \lambda_2} \frac{\partial U}{\partial I_1} + \left(\frac{\lambda_1}{\lambda_2} + \frac{\lambda_2}{\lambda_1} \right) \frac{\partial U}{\partial I_2} + \lambda_1 \lambda_2 \frac{\partial U}{\partial I_3}$$

For this nonlinear case there is no a simple expression for $\tau_{11}/\epsilon_{11} = E \quad -\epsilon_{22}/\epsilon_{11} = \nu$ (variable) even for this simple case. It is necessary to measure





and try to approximate the expressions of $\partial U / \partial I_i$

The reduction of these equations to those of a linear elastic solid in case $\lambda_i \to 1$ and U = quadratic function of I_1, I_2, is treated in problem 5, set 5

Incompressible material

$$\lambda_1 \lambda_2 \lambda_3 = 1 \qquad \therefore \quad I_3 = 1 = \text{constant}$$

$U = f(I_1, I_2)$ and equations symplify appreciatly.

The search of $U = f(I_1, I_2)$ can be done by the method of multipliers of Lagrange applying the principle of virtual work

$$\delta \left[W_e - U + p(\lambda_1 \lambda_2 \lambda_3 - 1) \right] = 0$$

$$(\lambda_2 \lambda_3)\tau_{11} = \frac{\partial U}{\partial \lambda_1} - \frac{p}{\lambda_1}$$

$$(\tau_1 \tau_3)\tau_{22} = \frac{\partial U}{\partial \lambda_2} - \frac{p}{\lambda_2}$$

etc

---

and finally $$\tau_{11} = 2\left(\lambda_1^2 - \frac{1}{\lambda_1}\right)\left(\frac{\partial U}{\partial I_1} + \frac{1}{\lambda_2}\frac{\partial U}{\partial \lambda_2}\right)$$

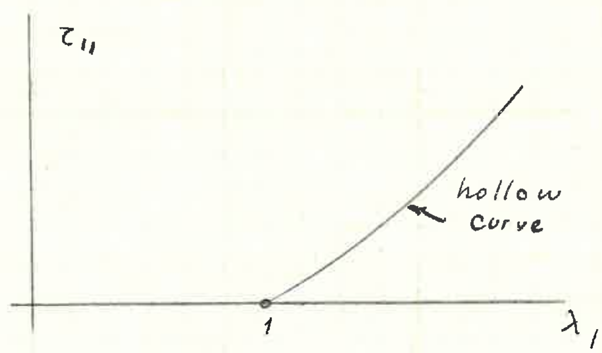
As a first approximation we might take

$$U = A\,(I_1 - 3)$$

(same form as those predicted by statistical theory of kinetic gases) and

$$\tau_{11} = 2A\left(\lambda_1^2 - \frac{1}{\lambda_1}\right)$$

constant A should be obtained from tests. This theory can be applied for the first part of the σ-ϵ curve for rubber type of materials as caoutchouc.



References:

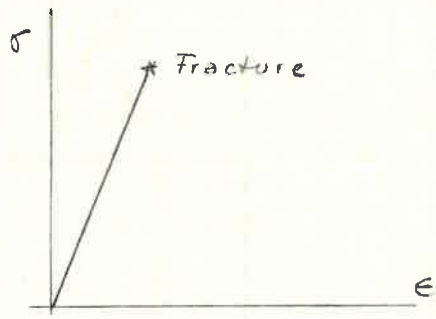
Mooney "A theory of Large elastic deformations" J. Appl. Physics sept 1940

Treloar "Rubber Problems"

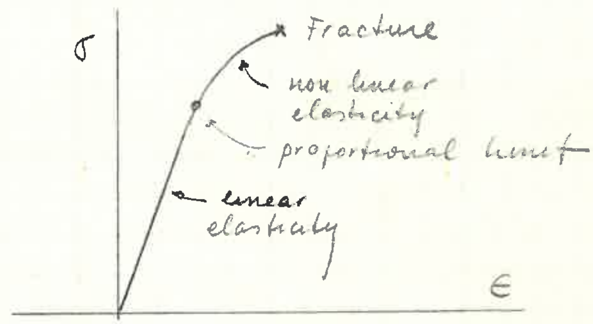
Green & Atkins "Non linear elasticity"

LIMITS OF APPLICATION OF ELASTIC THEORY

Some materials fail showing little or no plastic behavior at all ;

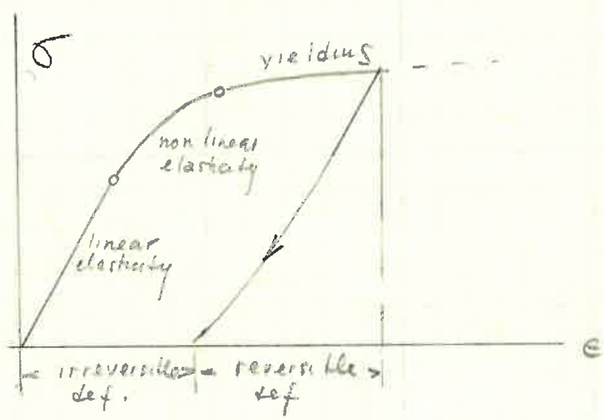


Linear elastic up to failure : glass



Fracture is preceded by non linear behavior

But many others materials show a pronounced plastic behavior before failure (some sort of "yielding")



Plastic behavior means that the material shows some amount of irrecoverable deformation after unloading (This generally depends on time)

The distinction between non-linear elastic and elastoplastic behavior is not easy to establish in practice, all depends on the precision of test machines.

Real materials usually show very complex behavior and the transition between one to another state is gradual and some characteristics combine. For instance

Mild steel	Hard steel	Concrete
Linear elasticity Short range of non linear elasticity Perfect yielding Strain-hardening	Linear elasticity Non " " Strain-hardening yielding	non linear elasticity + plasticity for all the σ-ε curve their ratio changing gradually.

For those materials having a definite plastic behavior the (practical) limit of elasticity will be called "yield point".

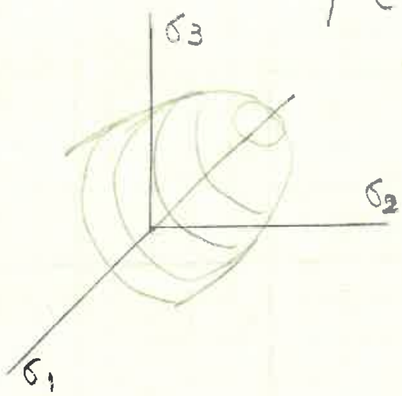
The YP is easily noticeable for materials as mild steel even depending on the rate of loading and for others as hard steel it is usually defined in a simple tension test as the stress which induces a definite irreversible deformation.

The main problem is: how to determine the YP or elastic limit for a more general state of stress.

Yield surface

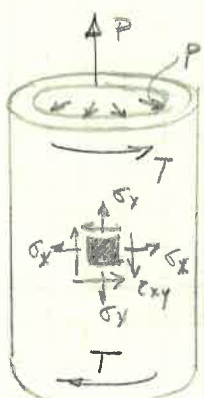
The condition of yielding for a tridimensional state of stress characterized by the principal stresses $\sigma_1, \sigma_2, \sigma_3$ can be stated as

$$f(\sigma_1, \sigma_2, \sigma_3) = 0$$



In a $(\sigma_1, \sigma_2, \sigma_3)$ reference system this equation represents the yield surface. If a point representing the state of stress lies inside the YS, $f(\sigma_1, \sigma_2, \sigma_3) < 0$ there is no yielding.

The construction of the YS is not a easy thing, even for materials with definite yield point. Usually the experiments on combined state of stress are performed using thin-walled cylinders



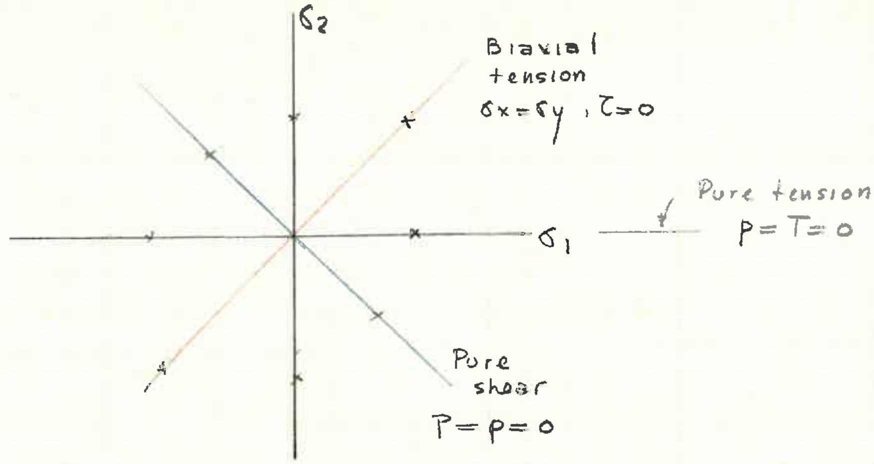
Applied: P axial force
p internal pressure
T torque

neglecting σ_z (produced by p) we are in a plane stress case

$$\sigma_x, \sigma_y, \tau_{xy}.$$

and we can get points of the curve

intersection of de YS with $\sigma_3 = \sigma_z \geq 0$



When a point in a body reaches the YS, we have a incipient plastic stress state. This does not mean in general the failure of the whole body (bases of ultimate design)

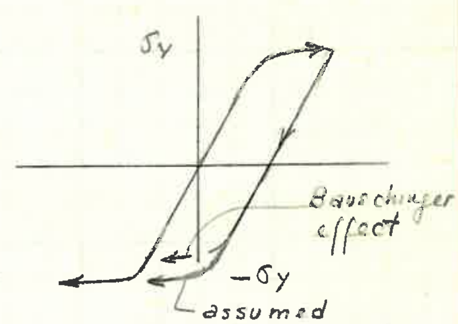
Properties

1. Y.S. depends in general upon loading-history (time) specially when strain-hardening occurs

2. For ductile metals, we can introduce some simplifications to reduce the complexity of the problem:

a. YS is symmetric respect to $\sigma_1 = \sigma_2 = \sigma_3$ (consequence of isotropy)

b. compression and tension tests are interchangeable (no Bauschinger effect as shown in the figure)



c. The change in volume during the plastic state is zero (incompressibility hypothesis) $\nu_{pl} = 1/2$.

This can be written $I_1 I_2 I_3 = 1$.

THEORIES OF FAILURE

For ductile* materials as many metals, there are two theories widely used

1. Maximum shear theory (Tresca, Guest, Mohr, etc)
2. " distortion energy theory (V. Mises, Hencky, Maxwell, Huber, Novozhilov, Beltrami)

* compression strength similar to tension strength

Maximum shear Theory (Tresca's criterion of yielding)

$$\frac{\sigma_{max} - \sigma_{min}}{2} = \tau_{yp}$$

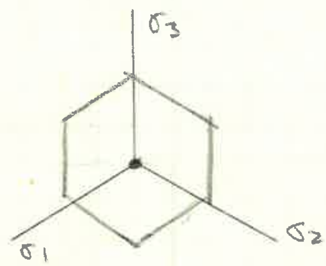
for simple tension test $\sigma_{max} - \sigma_{min} = \sigma_{yp}$ $\therefore$ $\sigma_{yp} = 2\tau_{yp}$

depending on the region:

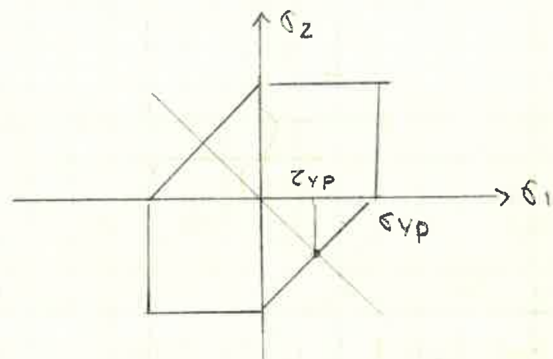
$$\sigma_1 - \sigma_2 = \pm\sigma_{yp} \qquad \sigma_1 - \sigma_3 = \pm\sigma_{yp} \qquad \sigma_2 - \sigma_3 = \pm\sigma_{yp}$$

also $(\sigma_1 - \sigma_2)^2(\sigma_2 - \sigma_3)^2(\sigma_3 - \sigma_1)^2 = \sigma_{yp}^2$

The YS is a regular hexagonal prism whose axis is $\sigma_1 = \sigma_2 = \sigma_3$



section for a plane $\perp$ to the axis $\sigma_1 = \sigma_2 = \sigma_3$ for instance $\sigma_1 + \sigma_2 + \sigma_3 = a$.



section $\sigma_3 = 0$ (plane stress)

Distortion Energy Theory (V. Mises, Maxwell, Huber, Hencky, Nadai, Eschinger, Novozhilov, Beltrami, etc, etc)

The yielding starts when the distortion elastic strain energy reaches a critical value

the total energy is $U = \frac{1}{2}\tau_{ij}\epsilon_{ij}$

set
$$\left.\begin{aligned} \tau_{ij} &= s_{ij} + \tfrac{1}{3}\Theta_1\delta_{ij} \\ \epsilon_{ij} &= e_{ij} + \tfrac{1}{3}\vartheta_1\delta_{ij} \end{aligned}\right\} \text{deviator + isotropic tensors}$$

$$U = \frac{1}{2}\left(s_{ij} + \frac{1}{3}\Theta_1\delta_{ij}\right)\left(e_{ij} + \frac{1}{3}\vartheta_1\delta_{ij}\right) =$$

$$= \frac{1}{2}s_{ij}e_{ij} + \frac{1}{6}s_{ij}\vartheta_1\delta_{ij} + \frac{1}{6}\Theta_1 e_{ij}\delta_{ij} + \frac{1}{18}\Theta_1\vartheta_1\delta_{ij}\delta_{ij}$$

but $\frac{1}{6}s_{ij}\vartheta_1\delta_{ij} = \frac{1}{6}s_{ii}\vartheta_1 = 0$

$$\frac{1}{6}\Theta_1 e_{ij}\delta_{ij} = \frac{1}{6}\Theta_1 e_{ii} = 0$$

$$\frac{1}{18}\Theta_1 \vartheta_1 \delta_{ij}\delta_{ij} = \frac{1}{18}(3\sigma_m)\vartheta_1 \cdot 3 = \frac{1}{2}\sigma_m \vartheta_1$$

$$\underline{U = \frac{1}{2} S_{ij} e_{ij} + \frac{1}{2}\sigma_m \vartheta_1} = \underline{U_d + U_h}$$

U_d = distortion strain energy U_h = hydrostatic strain energy.

The V. Mises yield criterion is

$$\boxed{U_d = \frac{1}{2} S_{ij} e_{ij} = \text{constant}}$$

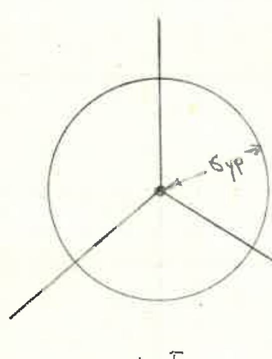
for a single tension test $U_d = \dfrac{\sigma_{yp}^2}{6\mu}$ then

$$U_d = \frac{1}{2} S_{ij} e_{ij} = \frac{\sigma_{yp}^2}{6\mu}$$

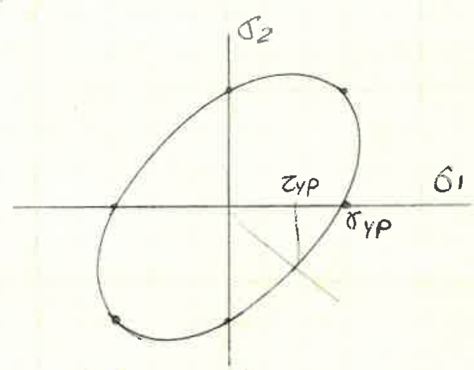
Other ways of writing it:

$$\frac{1}{2} S_{ij} S_{ij} = \frac{\sigma_{yp}^2}{3} \qquad \left(\text{because } e_{ij} = \frac{1}{2\mu} S_{ij}\right)$$

$$(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2 = 2\sigma_{yp}^2$$ in terms of principal stresses. From this we can easily seen that the yield surface $f(\sigma_1, \sigma_2, \sigma_3)$ is a circular cylinder whose axis is $\sigma_1 = \sigma_2 = \sigma_3$.



Section $\perp [\sigma_1 = \sigma_2 = \sigma_3]$



Section $\sigma_3 = 0$

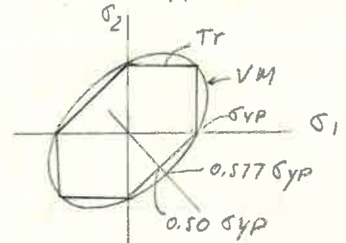
For simple shear we obtain $\tau_{yp} = \frac{1}{\sqrt{3}}\sigma_{yp}$ $\quad \sigma_{yp} = \sqrt{3}\,\tau_{yp}$

If Tresca and V. Mises criteria agree on σ_{yp}, the hexagon is inscribed in the circle. If they agree for τ_{yp}, the " circumscribes " " .

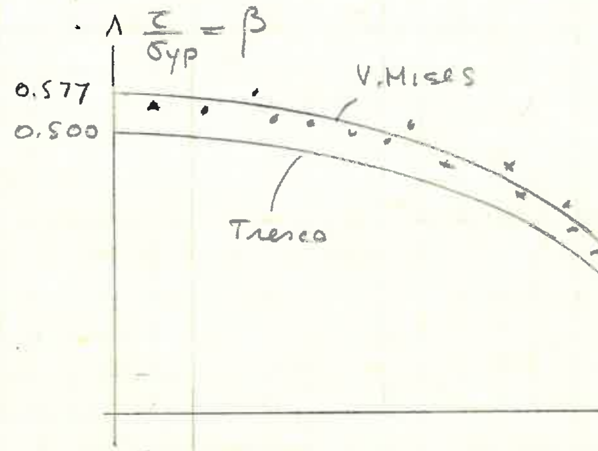
Comparison between the two yielding criteria

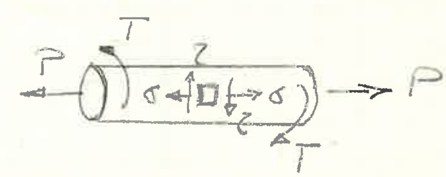
If both criteria agree for σ_{yp}. The maximum difference is found for simple shear

Tresca $\tau_{yp} = 0.500\ \sigma_{yp}$
V. Mises $\tau_{yp} = 0.577\ \sigma_{yp}$. } Dif 15.7%



Test run by   for ductile metals using thin cylinders (see page   ) show results closer to the V. Mises theory.



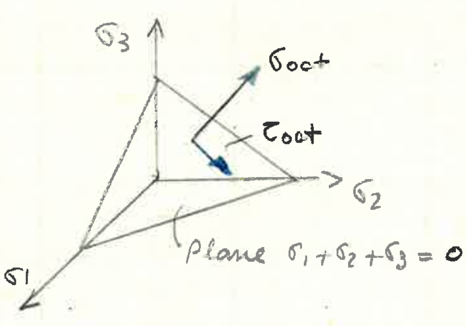


Tresca $\alpha^2 + 4\beta^2 = 1$
V Mises $\alpha^2 + 3\beta^2 = 1$

see Hoffmann & Sack "Plasticity"

Other ways of formulating the distortion energy theory

1. Octahedral stresses (Nadai)



Normal stress $\sigma_{oct} = \frac{1}{3}(\sigma_1+\sigma_2+\sigma_3) = \sigma_m$

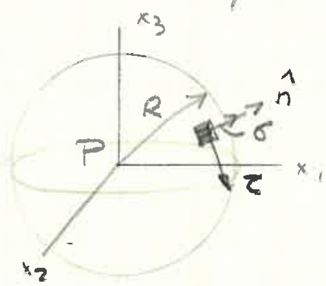
Shearing stress $\tau_{oct} = \frac{1}{3}\left[(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2\right]^{1/2}$

Nadai stated: yielding occurs when $|\tau_{oct}|$ reaches a critical value

$$|\tau_{oct}| = c = \sqrt{\frac{2}{3}}\,\sigma_{yp}$$

it is completely equivalent to the distorsion energy formulation $U_d = K$

2. Mean (quadratic) shear stress (Novozhilov)



Let a sphere of radius R and center P

Define the root mean square shear stress at P as

$$\tau_m = \lim_{S \to 0} \left[\frac{1}{S} \int_S \tau^2 dS \right]$$

and it turns out to be independent of R and equal to

$$\tau_m = \frac{1}{15} \left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right]^{1/2} = \frac{\tau_{oct}}{5}$$

$$\sigma_m = \frac{1}{3} (\sigma_1 + \sigma_2 + \sigma_3)$$

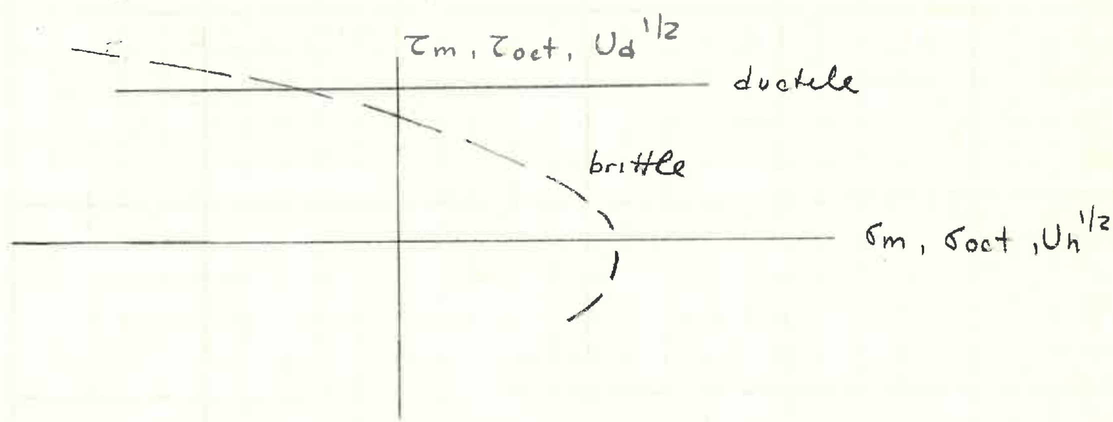
The formulation τ_m = critical value is entirely equivalent to the distortion energy criterion as we can easily see.

BRITTLE MATERIALS

For brittle materials this two theories do not apply - To study the Mohr theory (which includes ~~the~~ Tresca's as a particular case) see

Nadai "Theory of Fracture and Flow" ch 17
Timoshenko "Strength of Materials" Vol II last ch.

A generalization of the V. Mises Theory for brittle materials can be done plotting the envelope for τ_m (or τ_{oct}, or $U_d^{1/2}$) versus σ_m (or σ_{oct}, or $U_h^{1/2}$)



Note: The envelope for Mohr's theory is plotted on the (τ, σ) coordinates and does not account for the intermediate principal stress σ_2

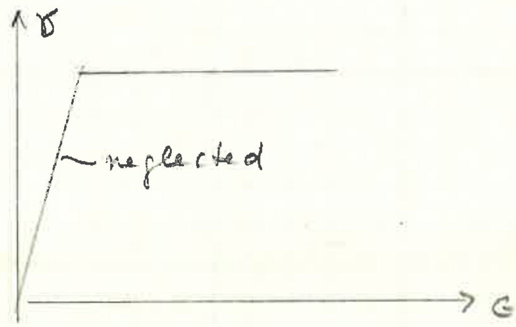
PERFECTLY PLASTIC SOLID THEORY

Characterized by

1. incompressible deformation (Volume = constan)
2. stress deviator is proportional to the strain rate (strain increment) deviator

$$S_{ij} = \tau_{ij} - \frac{1}{3}\delta_{ij}\tau_{kk} = \tau_{ij} - \sigma_m \delta_{ij}$$

ϵ''_{ij} = plastic strain tensor



The plastic strain deviator is

$$e''_{ij} = \epsilon''_{ij} - \frac{1}{3}\delta_{ij}\epsilon''_{kk}$$

but $\epsilon''_{kk} = \epsilon''_{11} + \epsilon''_{22} + \epsilon''_{33} = 0$ for postulate (1)

$$\Rightarrow \boxed{e''_{ij} = \epsilon''_{ij}}$$

The strain-rate deviator is

$$\dot{e}''_{ij} = \frac{d\epsilon''_{ij}}{dt}$$

where t is a parameter (This case must not be confused with viscoelasticity, because the perfect plastic solid is a inviscid solid. Rate means increment)

3. The coefficient of proportionality φ is a scalar invariant of the strain-rate deviator.

$$S_{ij} = \varphi\,\dot{e}''_{ij} = \varphi\,\dot{\epsilon}''_{ij}$$

If we use V. Mises criterion

$$J_2 = \frac{1}{2} S_{ij} S_{ij} = K^2$$

$$\frac{1}{2}(\varphi\epsilon''_{ij})(\varphi\cdot\epsilon''_{ij}) = K^2$$

$$\varphi^2 = \frac{2K^2}{\dot\epsilon''_{ij}\,\dot\epsilon''_{ij}} \qquad \varphi = \frac{\sqrt{2}\,K}{\sqrt{\dot\epsilon''_{ij}\,\dot\epsilon''_{ij}}}$$

$$S_{ij} = \frac{\sqrt{2}\cdot K}{\sqrt{\dot\epsilon''_{ij}\,\dot\epsilon''_{ij}}}\;\dot\epsilon''_{ij}$$

which is the Levy - St Venant - Mises equation (the first two had written it using Tresca's criterion)

Referred to the principal stress axes:

$$S_{11} = \frac{2\sigma_1 - \sigma_2 - \sigma_3}{2} \quad \text{etc}$$

$$2\sigma_1 - \sigma_2 - \sigma_3 = 3\varphi\,\dot\epsilon''_{11}$$

$$2\sigma_2 - \sigma_1 - \sigma_3 = 3\varphi\,\dot\epsilon''_{22}$$

$$2\sigma_3 - \sigma_1 - \sigma_2 = 3\varphi\,\dot\epsilon''_{33}$$

usually written in this way:

$$(2\sigma_1 - \sigma_2 - \sigma_3) : (2\sigma_2 - \sigma_1 - \sigma_3) : (2\sigma_3 - \sigma_1 - \sigma_2) = \dot\epsilon''_{11} : \dot\epsilon''_{22} : \dot\epsilon''_{33}$$

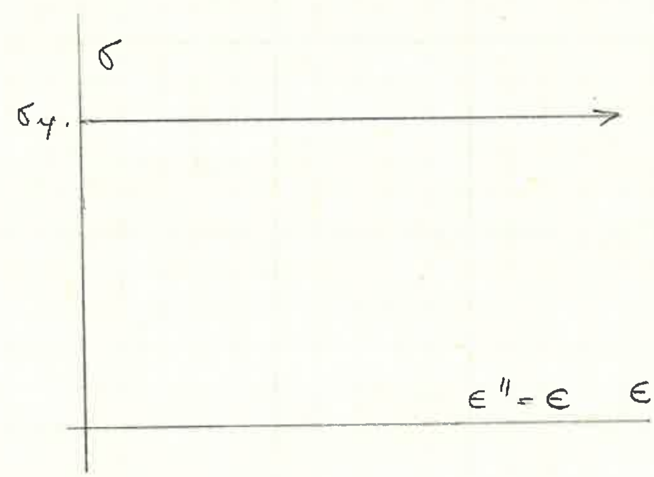
besides

$$d\epsilon''_{11} + d\epsilon''_{22} + d\epsilon''_{33} = \dot\epsilon''_{11} + \epsilon''_{22} + \epsilon''_{33} = 0$$

(volume constancy)

These are the constitutive equations for a perfectly plastic solid.

For $J_2 < K^2$ the body is <u>rigid</u>

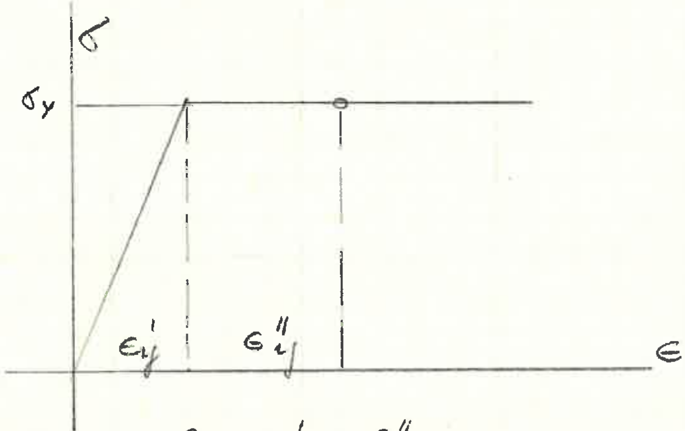


and this is call the Rigid - Perfectly - Plastic Solid Theory.

ELASTIC-PERFECTLY PLASTIC SOLIDS

The total strain is the sum of

ϵ'_{ij} = elastic strain
ϵ''_{ij} = plastic strain



$$\epsilon_{ij} = \epsilon'_{ij} + \epsilon''_{ij}$$

<u>Elastic zone</u>:

$$s_{ij} = 2\mu e'_{ij}$$
$$\sigma_{kk} = \left(\lambda + \frac{2\mu}{3}\right)\epsilon'_{kk}$$

(Hooke's law)

<u>Plastic zone</u> The plastic strain rate is determined by V. Mises' theory:

$$\dot{s}_{ij} = \varphi \dot{\epsilon}''_{ij}$$

The yield condition is $\frac{1}{2} s_{ij} s_{ij} = k^2$

Summing up, we have

$$e_{ij} = \epsilon_{ij} - \frac{1}{3}\delta_{ij}\epsilon_{kk}$$
$$e'_{ij} = \epsilon'_{ij} - \frac{1}{3}\delta_{ij}\epsilon'_{kk}$$
$$e''_{ij} = \epsilon''_{ij} - \frac{1}{3}\delta_{ij}\epsilon''_{kk}$$

and

$$\dot{e}_{ij} = \dot{e}'_{ij} + \dot{e}''_{ij}$$

$$\boxed{\dot{e}_{ij} = \frac{\dot{s}_{ij}}{2\mu} + \frac{s_{ij}}{\varphi}}$$

Let's call $\lambda = 1/\varphi$

$$\dot{e}_{ij} = \frac{\dot{s}_{ij}}{2\mu} + \lambda s_{ij}$$

$$\boxed{\dot{s}_{ij} = 2\mu\left(\dot{e}_{ij} - \lambda s_{ij}\right)}$$

which is the Prandtl-Reuss equation. Note that λ is not a material parameter, it depends on the history of the loading (it's a loading parameter). Conversely, μ is a parameter.

Multiply now both sides by s_{ij} and sum:

$$\dot{s}_{ij} s_{ij} = 2\mu\left(\dot{e}_{ij} s_{ij} - \lambda s_{ij} s_{ij}\right)$$

but $\quad s_{ij} s_{ij} = 2k^2$ from the yield condition

and $\quad \dot{s}_{ij} s_{ij} = \frac{d}{dt}\left(s_{ij} s_{ij}\right) = 0$

Then

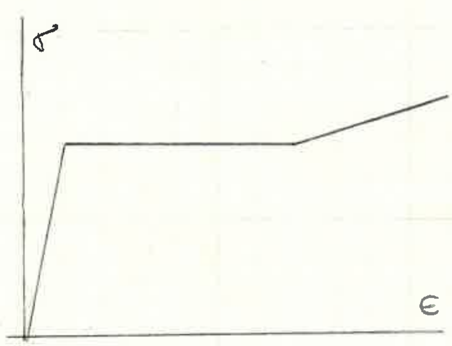
$$0 = 2\mu\left(\dot{e}_{ij} s_{ij} - 2\lambda k^2\right)$$

$$\boxed{\lambda = \frac{s_{ij}\dot{e}_{ij}}{2k^2}}$$

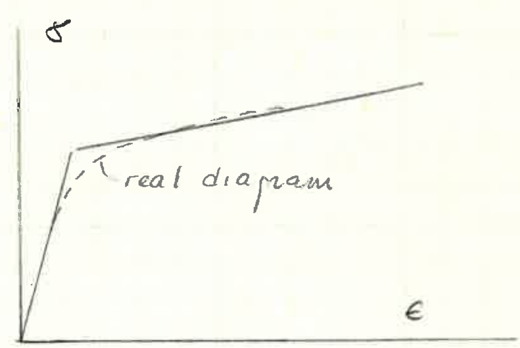
The term $s_{ij}\dot{e}_{ij}$ is the rate of work by deviatoric stresses

STRAIN-HARDENING PLASTIC MATERIALS

Typical σ-ϵ diagrams of SH materials are:

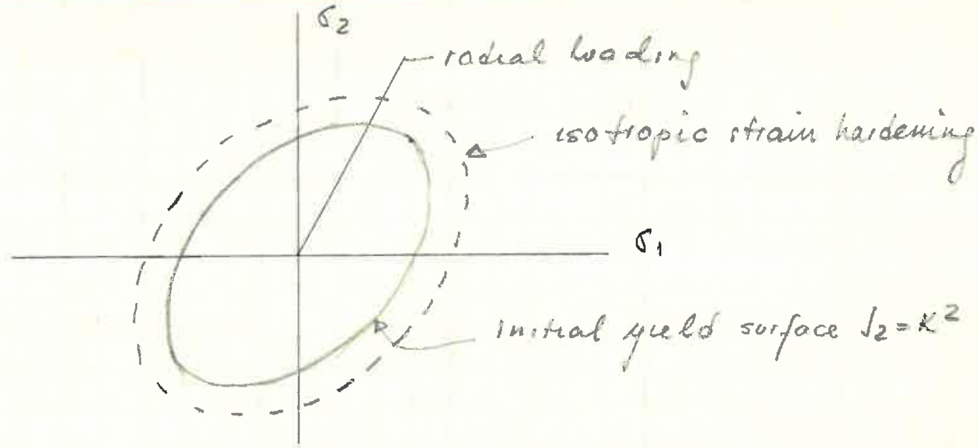


Mild steel



Hard steel
Aluminum alloys, etc

The fundamental problem for the analysis of this case is to find a tridimensional criterion of "yielding", because the yielding surfaces usually change with the loading history



For the case of radial loading (stresses maintain constant ratio) the yield surface expands uniformly once the SH has started. But for the general case of loading - unloading or no radial loading, the YS change its shape (Bauschinger effect) and that would mean we have to perform an experiment for each case. The theory becomes then terrible complex.

General Properties of the Yield Surface

Note: to study the "stability postulate" of Drucker, see

Proceeding UC Congress of Applied Mechanics 1, 3, 4' Congress
Simposium on Naval Structural Mechanics 1960-61 Standford U
Journal of Applied Mechanics 1950-53.

1. Yield Surface is convex
2. Plastic strain increment is normal to the YS

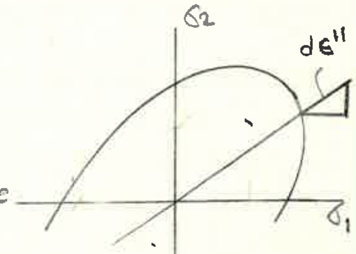
$$\dot{\epsilon}''_{ij} = \Lambda \frac{\partial f}{\partial S_{ij}} \qquad (f = 0 \text{ is the YS equation})$$

Λ is a scalar function which may depend on
strain
strain history
strain hardening

For the V. Mises yield function (perfect plastic solids):

$$f = J_3 - K^2 = \frac{1}{2} S_{ij} S_{ij} - K^2$$

$$\frac{\partial f}{\partial S_{ij}} = S_{ij} \quad \therefore \quad \dot{\epsilon}''_{ij} = \Lambda S_{ij}$$



for example for $\sigma_3 = 0$ $\quad \dot{\epsilon}''_{11} = \Lambda S_{11} = \Lambda \frac{1}{3}(2\sigma_1 - \sigma_2)$ etc.

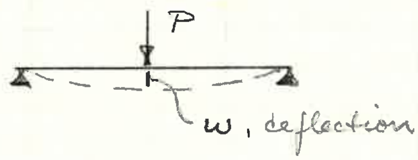
$$\dot{\epsilon}''_{11} : \dot{\epsilon}''_{22} = (2\sigma_1 - \sigma_2) : (2\sigma_2 - \sigma_1)$$

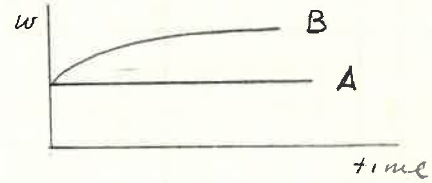
LINEAR VISCOELASTIC SOLID

A viscoelastic solid is a material for which <u>time</u> plays a fundamental role in its stress-strain relationship.

A material whose constitutive equations does not depend on time is said to be an <u>inviscid</u> solid

Example



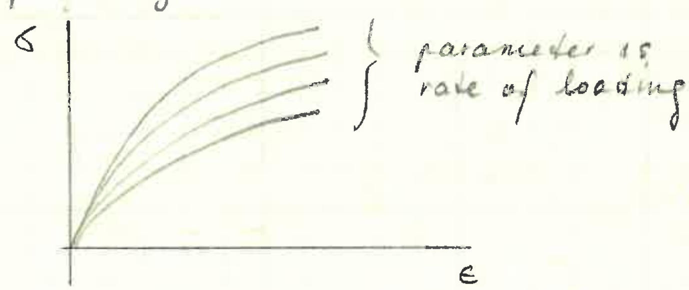


A is an inviscid material
B is a viscid (viscoelastic), time-dependent material (capable of creep)

The behavior of a solid in creep depends specially on

temperature (very important for most of them)
stress level.

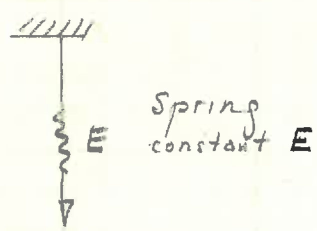
If we perform a tension test we will find different curves depending on rate of loading



<u>Models</u>

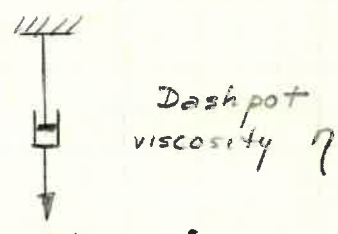
The behavior of real viscoelastic material can be <u>approximate</u> by some <u>ideal</u> models, which are quite useful to understand and predict responses

They are <u>not</u> related to material structure, they are conceptual elements.



$\sigma = E\epsilon$

Linear element or Hooke element

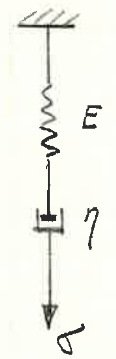


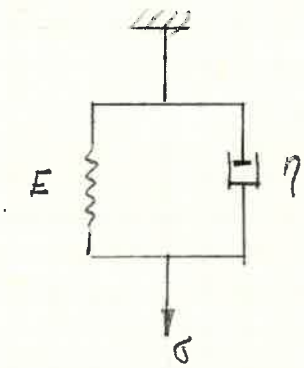
$\sigma = \eta \frac{d\epsilon}{dt} = \eta \dot{\epsilon}$

Linear Viscous or Newtonian element

These two basic models can be combined to form more complex models. There are two "two-elements" model :

Maxwell model





Kelvin or Voigt model

Maxwell model : spring + dashpot in series : <u>common stress</u> σ

$$\epsilon(t) = \epsilon' + \epsilon''$$

$$\dot{\epsilon} = \dot{\epsilon}' + \dot{\epsilon}'' = \frac{\dot{\sigma}}{E} + \frac{\sigma}{\eta} \quad \text{or}$$

$$\boxed{\frac{d\epsilon}{dt} = \frac{1}{E}\frac{d\sigma}{dt} + \frac{\sigma}{\eta}}$$

$$\dot{\sigma} + \frac{E}{\eta}\sigma = E\dot{\epsilon}$$

Kelvin model : spring + dashpot in paralel : <u>common strain</u> ϵ

$$\sigma(t) = \sigma' + \sigma''$$

$$\sigma = E\epsilon + \eta\dot{\epsilon} \quad \text{or}$$

$$\boxed{\sigma = E\epsilon + \eta\frac{d\epsilon}{dt}}$$

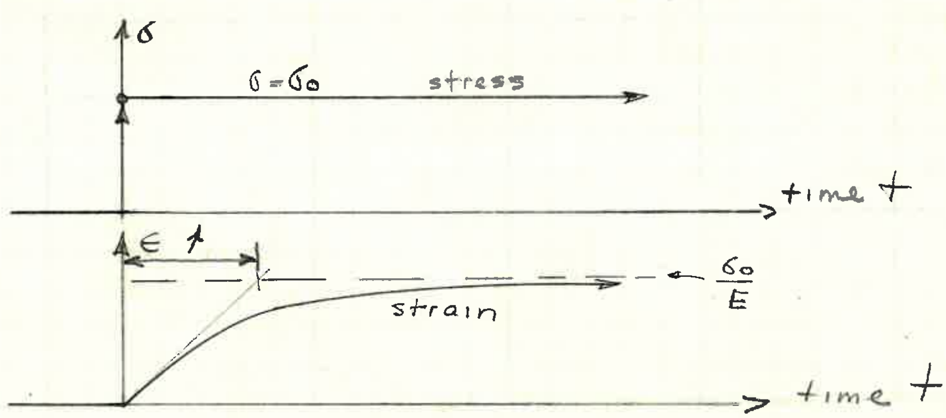
$$\dot{\epsilon} + \frac{E}{\eta}\epsilon = \frac{1}{\eta}\sigma$$

<u>Creep Experiment</u> : a creep experiment consist of applying a force or stress σ to a model and then maintaining this stress constant the extension or strain ϵ is measured as a function of the time. Taking $t = 0$ as the instant of loading, the stress imposed is

$$\sigma = \sigma_0 H(t)$$

where $H(t)$ is the Heaviside unit step function.

The <u>Kelvin</u> model is a very useful one to study creep response because it gives an <u>asymptotic</u> creep curve as many real materials



Integrating the Kelvin model equation:

$$\eta \frac{d\epsilon}{dt} + E\epsilon = \sigma = \sigma_0 H(t) \quad \text{with } \epsilon = 0 \text{ for } t = 0$$

we find $\epsilon = \frac{\sigma_0}{E}\left(1 - e^{-\frac{t}{\tau}}\right) H(t)$ as $t \to \infty$, $\epsilon \to \frac{\sigma_0}{E}$

where $\tau = \frac{\eta}{E}$ is called the <u>retardation time</u>. The meaning of this constant can be seen in the figure (precedent page). In fact the equation of the initial tangent is $\epsilon = (\sigma_0/\eta)\, t$ which intercepts the asymptotic line $\epsilon = \sigma_0/E$ at $t = E/\eta = \tau$

The <u>creep compliance</u> $J(t)$ is defined as

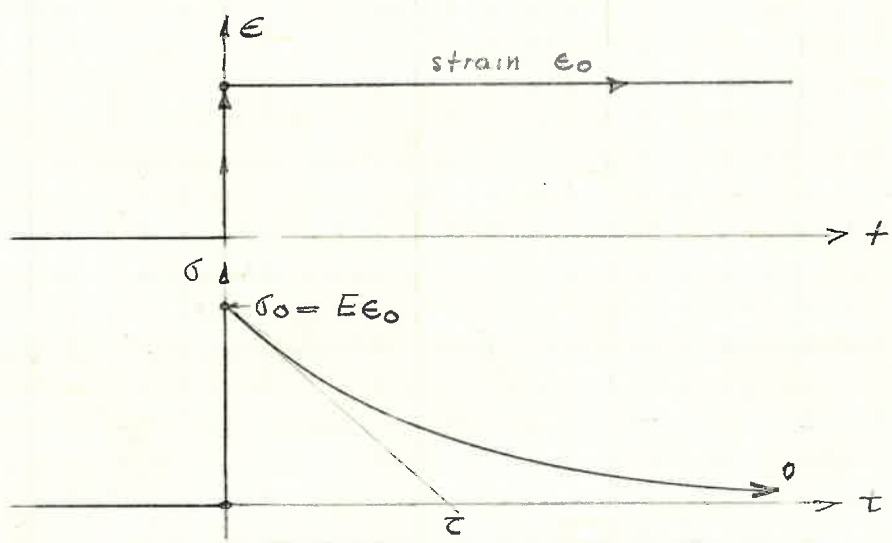
$$\epsilon = \sigma_0 \, J(t)$$

in this case $J = \frac{1}{E}\left(1 - e^{-\frac{t}{\tau}}\right)$

Stress Relaxation Experiment

A relaxation experiment consist of imposing a constant extension $\epsilon = \epsilon_0 H(t)$ and measuring the force or stress σ required to do so as a function of the time.

The Maxwell model is useful to describe a relaxation process



From

$$\frac{1}{E}\frac{d\sigma}{dt} + \frac{\sigma}{\eta} = \frac{d\epsilon}{dt} = \epsilon_0 \, \delta(t)$$

where $\delta(t)$ is the delta function $\frac{dH(t)}{dt}$ ($\delta(t) = 0$ for all t except $\delta(t) = \infty$ for $t = 0$)

and the condition $\sigma = E\epsilon_0 = \sigma_0$ for $t=0$ we get

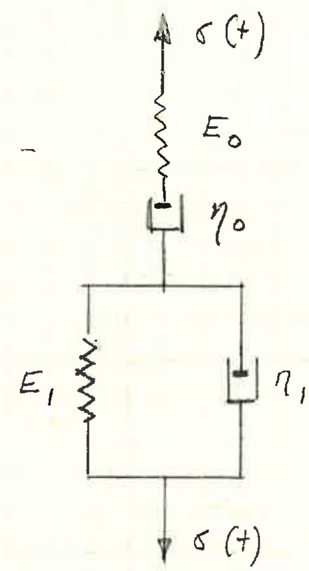
$$\sigma = \sigma_0 \cdot e^{-\frac{t}{\tau}} \, H(t)$$

as $t \to \infty$
$\sigma \to 0$

where $\tau = (\eta / E)$ is the relaxation time (see meaning in figure)

Burgers Model

A particularly useful model is that resulting of combining a Maxwell and a Kelvin in series (it is one type of four-elements model)



$$\epsilon = \epsilon_0 + \epsilon_0' + \epsilon_1$$

ϵ_0 = extension of the spring E_0
ϵ_0' " " dashpot η_0
ϵ_1 " " Kelvin model

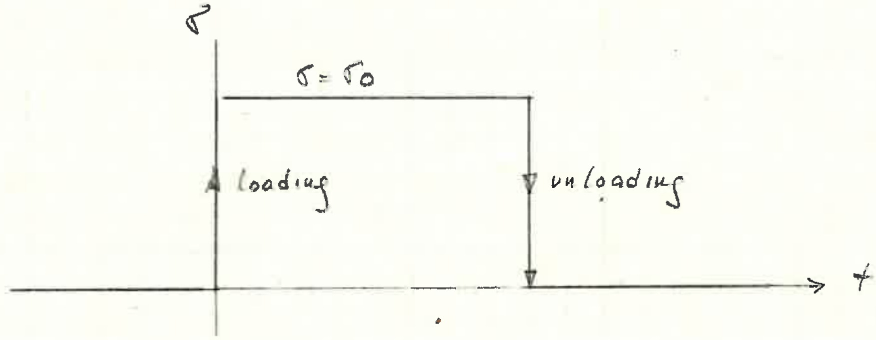
Now
$$\sigma = E_0 \epsilon_0$$
$$\sigma = \eta_0 \dot{\epsilon}_0'$$
$$\sigma = E_1 \epsilon_1 + \eta_1 \dot{\epsilon}_1$$

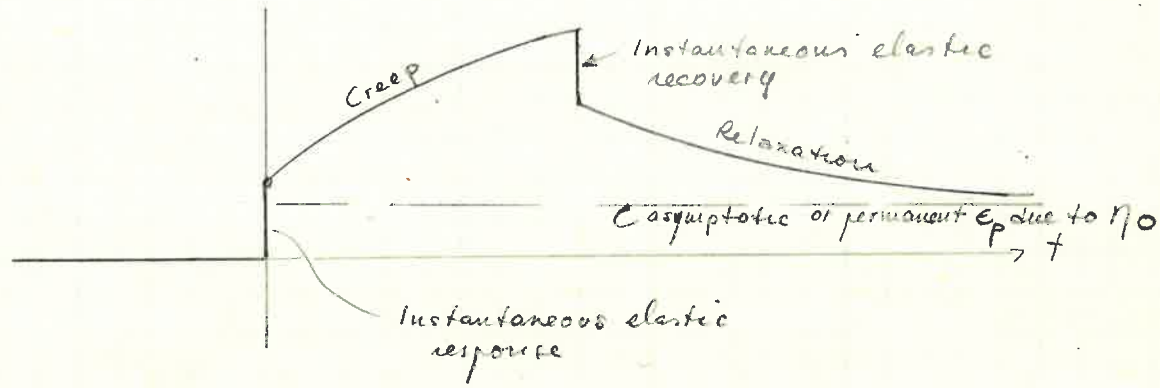
substituting we get

$$\frac{d^2\sigma}{dt^2} + \left(\frac{E_0}{\eta_0} + \frac{E_0}{\eta_1} + \frac{E_1}{\eta_1} \right) \frac{d\sigma}{dt} + \frac{E_0 E_1}{\eta_0 \eta_1} \sigma = E_0 \frac{d^2\epsilon}{dt^2} + \frac{E_0 E_1}{\eta_1} \frac{d\epsilon}{dt}$$

, σ-ϵ relationship for the Burgers model.

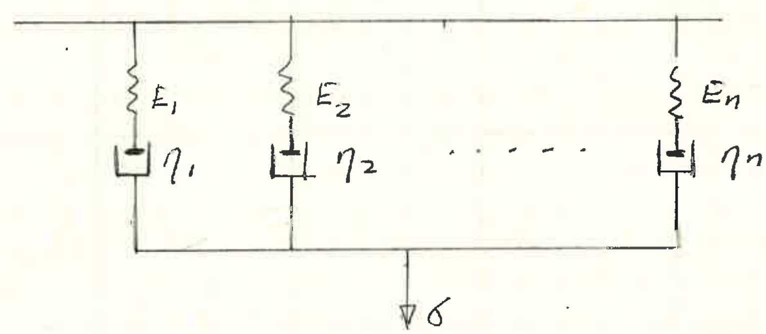
With this model one may approximate behavior of some real materials:





For strictly linear solids $\epsilon_p = 0$ ...... (complete recovery).

<u>Generalized Maxwell Model</u> (Maxwell models in parallel)



ϵ and $\dot{\epsilon}$ constant for all elements

$$\sigma = \sigma_1 + \sigma_2 + \cdots + \sigma_n$$

$$\dot{\epsilon}_i = \frac{1}{E_i}\dot{\sigma}_i + \frac{1}{\eta_i}\sigma_i = \left(\frac{p}{E_i} + \frac{1}{\eta_i}\right)\sigma_i = p\,\epsilon_i$$

where p is the operator $\frac{d}{dt}$

$$\left(p + \frac{1}{\tau_i}\right)\sigma_i = p E_i \epsilon \quad \therefore \quad \sigma_i = \frac{p E_i \epsilon}{p + \frac{1}{\tau_i}}$$

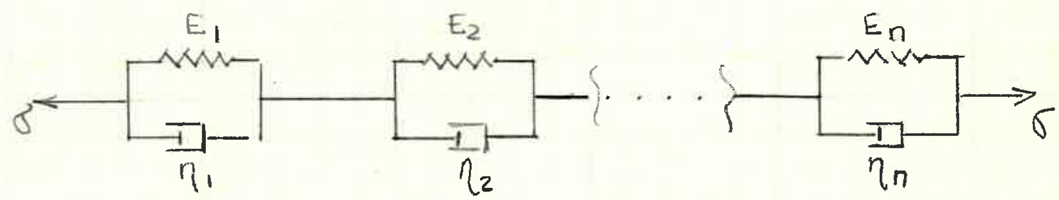
$\tau_i = \eta_i / E_i$ = relaxation time

$$\sigma = \sum_{i=1}^{n} \left[\frac{p E_i}{p + \frac{1}{\tau_i}} \right] \epsilon(t)$$

Material parameters $E_1, E_2, \dots E_n$
$\tau_1, \tau_2 \dots \tau_n$

This model may fit many experimental relaxation curves using convenient number of elements. This is usually done applying a extension or <u>input</u> $\epsilon = \epsilon(t)$ and measuring the stress response $\sigma = \sigma(t)$ or <u>output</u>. For this purpose, very complex and costly test machines are used in the labs of great industrial companies to study the new materials.

<u>Generalized Kelvin Model</u> (Kelvin models in series)



$$\sigma = E_i \epsilon_i + \eta \dot{\epsilon}_i = (E + \eta p)\epsilon_i \quad \therefore \quad \epsilon_i = \frac{\sigma}{E_i + \eta_i p}$$

$$\epsilon = \sum_{i=1}^{n} \epsilon_i = \sum_{i=1}^{n} \left[\frac{1}{E_i + \eta_i p} \right] \sigma(t)$$

because σ is the same for each element. This may fit creep experimental curves.

<u>General Differential Constitutive laws</u>

Using the measured viscoelastic properties, and following the approach suggested by the ideal model behavior, we can write the general constitutive laws for a linear viscoelastic solid in a differential way (for uniaxial stress-strain)

$$a_0 \sigma + a_1 \frac{d\sigma}{dt} + a_2 \frac{d^2\sigma}{dt^2} + \dots + a_n \frac{d^n \sigma}{dt^n} =$$

$$= b_0 \epsilon + b_1 \frac{d\epsilon}{dt} + b_2 \frac{d^2\epsilon}{dt^2} + \dots + b_m \frac{d^m \sigma}{dt^m}$$

$$\sum_{k=1}^{n} [a_k p^k]\, \sigma = \sum_{j=0}^{m} [b_j p^j]\, \epsilon$$

$$P\sigma = Q\epsilon$$

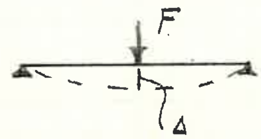
where P and Q are linear differential operators.

$$\sigma = \frac{Q}{P}\epsilon = \text{"}E\text{"}\epsilon \qquad \epsilon = \frac{P}{Q}\sigma = \text{"}J\text{"}\sigma$$

$(Q/P) \longleftrightarrow E$ modulus of elasticity for elastic solids
$(P/Q) \longleftrightarrow J$ elastic compliance

are generalizations of the concept of E, J for elastic solids

, example :

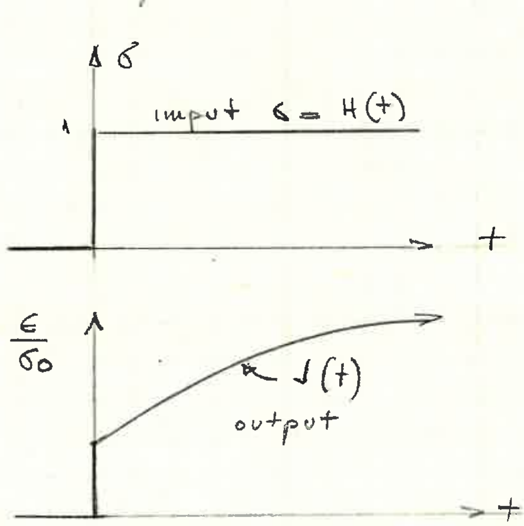


$$\Delta = \frac{Fl^3}{48EI} \quad \text{elastic case}$$

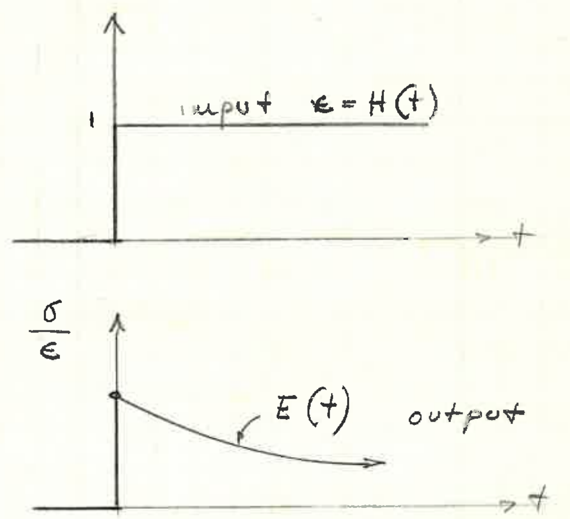
$$\Delta(t) = \frac{Fl^3}{48\left(\frac{Q}{P}\right)I} \quad \text{viscoelastic case } f(\text{time})$$

This is the approach using differential operators. But there is another and more direct approach, using the experimental creep and relaxation curves directly and applying integral formulas to get σ, ϵ for any input. This case will be treated now.

Definitions :



Unit creep curve or creep compliance curve : response to a unit stress input



Unit relaxation curve or relaxation modulus curve : response to a unit strain input

Integral Laws

The unit creep curve known, we want to compute the strain ϵ corresponding to any input $\sigma = \sigma(t)$. The solution is based on :

1. Linearity : we assume linear viscoelasticity, i.e. The compliance does not depend on σ or ϵ (If input is multiplied by α, output is also multiplied by α)

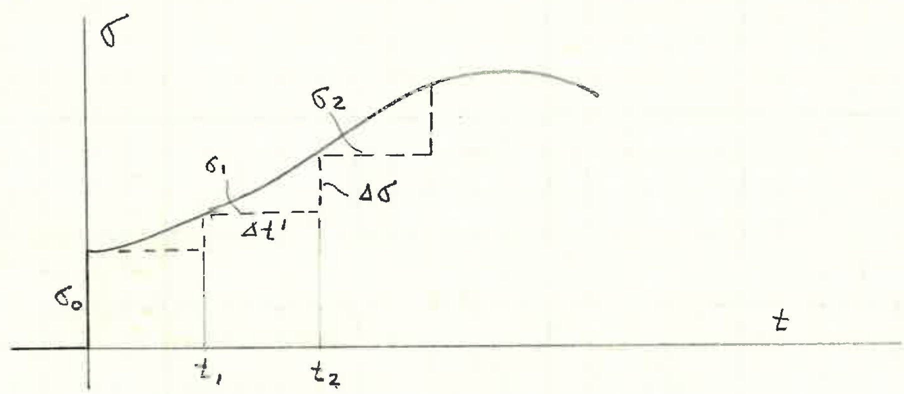
2. Simple heredity principle (Volterra) also called Boltzmann's Superposition principle. material properties are independent of absolute time (no aging)

The $\left\{ \begin{matrix} \text{strain} \\ \text{stress} \end{matrix} \right\}$ at the time t due to the application of several $\left\{ \begin{matrix} \text{stresses} \\ \text{strains} \end{matrix} \right\}$ in sucession is the summation of the $\left\{ \begin{matrix} \text{strains} \\ \text{stresses} \end{matrix} \right\}$ that would occur at that time if each $\left\{ \begin{matrix} \text{stress} \\ \text{strain} \end{matrix} \right\}$ had been applied independently.

Note here : both principles are not rigurously valid for a material like concrete which shows non-linear creep behavior (for high stress level) and whose properties change with time → aging effect (specially for young concrete)

Creep integral law

The input $\sigma = \sigma(t)$



is approximated by a "multiple step" function as shown. The creep response is then

$$\epsilon(t) = \sigma_0 J(t) + \sigma_1 J(t-t_1) + \sigma_2 J(t-t_2) + \ldots$$

$$= \sigma_0 J(t) + \sum_{t'=\Delta t'}^{t} \frac{\Delta\sigma}{\Delta t'} J(t-t') \Delta t'$$

Equivalence of both integral laws

Using Laplace transform

$$\mathcal{L}\left[F(t)\right] = F^*(p) = \int_{0+}^{\infty} F(t)\, e^{-pt}\, dt$$

From creep law $\quad \epsilon^*(p) = \cancel{\sigma(0) J^*(p)} + J^*(p)\left[p\sigma^* - \cancel{\sigma(0^+)}\right]$

$$= p\,\sigma^* J^*$$

From relaxation law $\quad \sigma^*(p) = p\,\epsilon^* E^*$

then $\quad \boxed{E^* J^* p^2 = 1}$

taking the inverse

$$\mathcal{L}^{-1}\left(E^* J^*\right) = \mathcal{L}^{-1}\left(\frac{1}{p^2}\right) = t$$

$$\boxed{\int_{0+}^{t} E(t-t')\, J(t)\, dt' = t}$$

This integral equation gives the relationship between creep compliance and relaxation modulus. If one is given, the other can be calculated, at least numerically.

Summary of Uniaxial Viscoelastic Laws

Creep

$$\epsilon(t) = \sigma_0 J(t) + \int_0^t J(t-t')\,\frac{d\sigma}{dt'}\,dt'$$

Relaxation

$$\sigma(t) = \epsilon_0 E(t) + \int_0^t E(t-t')\,\frac{d\epsilon}{dt'}\,dt'$$

Creep and relaxation laws are not independent. There is a functional relation between $E(t)$ and $J(t)$:

$$E^* J^* = 1/p^2 \qquad \text{(Laplace Transf. Law)}$$

This results should be compared with those for a elastic material:

$$\epsilon = \sigma J$$

$$\sigma = \epsilon E$$

$$E J = 1$$

Three Dimensional Viscoelasticity Laws

Let us split

$$\epsilon_{ij} = e_{ij} + \delta_{ij}\,\epsilon_m$$
$$\tau_{ij} = S_{ij} + \delta_{ij}\,\sigma_m$$

e_{ij}, S_{ij} = deviator tensors and ϵ_m, σ_m : mean strain and stress respectively :

$$\sigma_m = \frac{1}{3}\tau_{kk} \qquad \epsilon_m = \frac{1}{3}\epsilon_{kk} \qquad \sigma_m = 3B\,\epsilon_m$$

B = Bulk Modulus.

We shall write the constitutive equation for elastic and VE solids together to compare

	Viscoelastic	Elastic
Relaxation	$S_{ij}(t) = e_{ij}(0)\,G_1(t) + \int_0^t G_1(t-t')\frac{\partial e_{ij}}{\partial t'}\,dt'$	$S_{ij} = 2\mu\, e_{ij}$
	$\sigma_m(t) = \epsilon_m(0)\,G_2(t) + \int_0^t G_2(t-t')\frac{\partial \epsilon_m}{\partial t'}\,dt'$	$\sigma_m = 3B\,\epsilon_m$
Creep	$e_{ij}(t) = S_{ij}(0)\,J_1(t) + \int_0^t J_1(t-t')\frac{\partial S_{ij}}{\partial t'}\,dt'$	$e_{ij} = \frac{1}{2\mu} S_{ij}$
	$\epsilon_m(t) = \sigma_m(0)\,J_2(t) + \int_0^t J_2(t-t')\frac{\partial \sigma_m}{\partial t'}\,dt'$	$\epsilon_m = \frac{1}{3B}\sigma_m$

G_2 Bulk Relaxation Modulus
J_2 " Compliance "

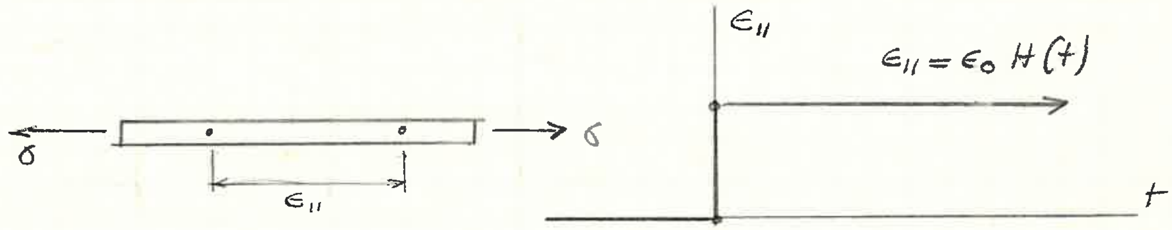
G_1 Shear Relaxation Modulus
J_1 " Compliance "

The purpose of writing the equations this way is that for many problems it is convenient to do so, for example : certain materials show no appreciable creep for volumetric stresses, but creep for deviatoric ones. In this case we use the elastic equation for ϵ_m and the VE one for e_{ij}

It is always possible to relate these constants to the more familiar concepts of elastic modulus $E(t)$ and Poisson ratio $\nu(t)$

Calculation of $E(t)$ & $\nu(t)$ from G and J functions

Assume a simple uniaxial relaxation test



$$\tau_{11} \neq 0 \qquad \tau_{22} = \tau_{33} = 0 \qquad \tau_{ij} = 0 \quad i \neq j$$

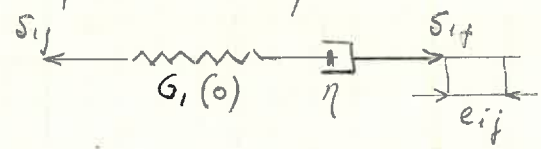
$$E(t) = \frac{\tau_{11}(t)}{\epsilon_0} \qquad \nu(t) = -\frac{\epsilon_{22}(t)}{\epsilon_0} = -\frac{\epsilon_{33}(t)}{\epsilon_0}$$

Substitute these data into the relaxation law and use Laplace Transform to eliminate ϵ, and we get for $E(t)$ and $\nu(t)$

$$E(t) = \mathcal{L}^{-1}\left[\frac{3 G_1^* G_2^*}{G_1^* + 2 G_2^*}\right]$$

$$\nu(t) = \mathcal{L}^{-1}\left[\frac{G_1^* - G_2^*}{p(G_1^* + 2G_2^*)}\right]$$

For a material which has an elastic Bulk Modulus B and a VE shear modulus of Maxwell type :



we get

$$E(t) = E_0 \exp\left[-\frac{2}{3}(1+\nu_0)\frac{t}{\tau}\right]$$

$$\nu(t) = \frac{1}{2}\left\{1 - (1-2\nu_0)\exp\left[-\frac{2}{3}(1+\nu_0)\frac{t}{\tau}\right]\right\}$$

where ν_0, E_0 are elastic moduli for $t = 0$ and $\tau = \eta / G_1(0)$

Note that for $t \to \infty$ $\quad E(t) \to 0 \quad \nu \to 1/2$

NOTE : During all this study we have dealt only with isotropic materials. To extends the VE law to an anisotropic solid we have to start with

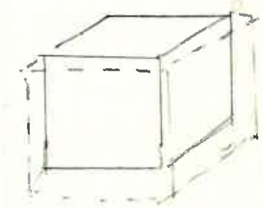
$$\tau_{ij} = C_{ijkl}\, \epsilon_{kl}$$

C = operational tensor, etc. Very little work has been done on this subject, which can be proved useful for many new reinforced and/or prestressed plastic — or even reinforced & prestressed concrete —

EFFECT OF TEMPERATURE IN STRESS-STRAIN LAWS

Thermally isotropic Materials

The coefficient of linear thermal expansion α is the same for all directions (Note: an elastic isotropic material may be thermally anisotropic)



$$\epsilon_{ij} = \alpha \, \delta_{ij} T$$

T change from a reference state where $\epsilon_{ij} = 0$
α mean thermal expansion coefficient for this range

Thermally Anisotropic Materials :

In this case we have an α_{ij} tensor of thermal expansion.

Thermoelastic isotropic stress-strain law

$$\epsilon_{ij} = \frac{1+\nu}{E} \tau_{ij} - \frac{\nu}{E} \Theta \delta_{ij} + \alpha \delta_{ij} T$$

let $i=j$ and sum

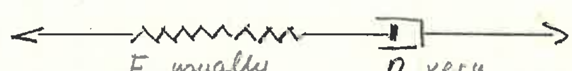
$$\vartheta_1 = \epsilon_{ii} = \frac{1+\nu}{E} \tau_{KK} - \frac{3\nu}{E} \Theta_1 + 3\alpha T$$

$$\vartheta_1 - 3\alpha T = \Theta_1 \left(\frac{1-2\nu}{E} \right)$$

Viscoelasticity

Wherever $\epsilon_m = \vartheta_1/3$ appears replace it by $\epsilon_m - \alpha T$

Until now we have assumed that the change T is so small that we do not have to consider a change in the mechanical properties. This can be more or less true for elasticity but it is not so approximate in viscoelasticity because the viscous terms usually depend highly on temperature changes. For instance

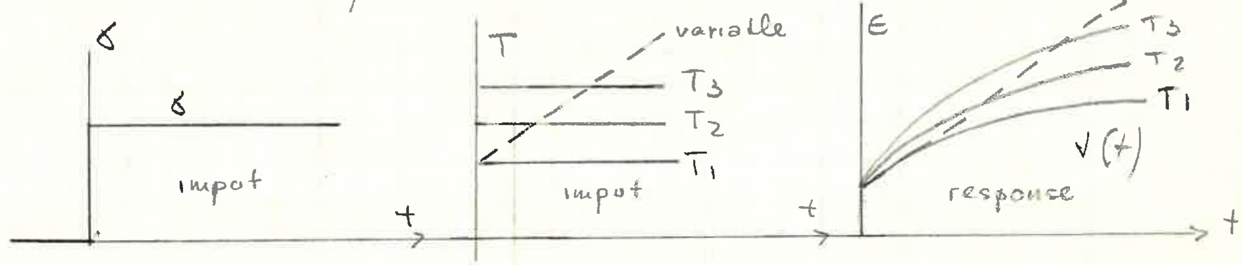


the ideal viscosity η being in many cases a exponential function of temperature $\eta = \eta_0 \exp(-KT)$

Material "constants" $J(t)$ $E(t)$ must be replaced then by temperature dependent functions

$$J(t,T) \qquad E(t,T)$$

To be obtained by tests



In case there is a gradient of T through the body (example thick wall cylinder under T_{int} and T_{ext}) the material behaves as an anisotropic body because $J(t, T)$ is different ~~at~~ each point.

Plastic Region.

The influence of T has not been well studied because of the complexity of the problem. T may induce a kind of viscoplastic (non-linear) behavior and the material may behave as a Kelvin or Maxwell model, all dependi on the relationships between mechanic and thermal properties.

FORMULATION OF PROBLEMS IN SOLID (LINEAR) MECHANICS

We have 1.) ; 3 equilibrium equations $\tau_{ij,j} + \rho f_i = \rho a_i$

2) 6 strain-displ. " $\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$

3) 6 compatibility " relating ϵ_{ij}

4) 6 constitutive " functions or functionals (VE) connecting stress and strain; whose characteristics vary with the kind of material we assume to deal with:

elastic = $f(\epsilon)$ time and loading-history indep.
VE $f(\epsilon$, time, loading history)
Plastic $f(\epsilon$, yielding criteria, rate of loading loading history)

5) Boundary conditions on S (surface)

a) Displacement body conditions $u_i = \overline{u_i}(S)$
displacement is prescribed on boundary

b) Stress body conditions $\tau_{ij} n_j = \overline{t_i}(S)$
stress vector is prescribed on boundary

for $\Delta t' \to 0$

$$\epsilon(t) = \sigma_0 J(t) + \int_{0^+}^{t} J(t-t') \frac{d\sigma}{dt'} dt'$$

This is called the Volterra Hereditary Stress-Strain Law or the Duhamel Integral Formulation

The main difficulty to get $J(t)$ from a test response to $\sigma = \sigma_0 H(t)$ is that any machine cannot induce a σ_0 instantaneously.

Another way of writing $\epsilon(t)$ is

$$\epsilon(t) = \sigma_0 J(t) + \int_{0^+}^{t} \sigma(t-t') \frac{dJ}{dt} dt'$$

which can be obtained writing $\epsilon(t) = \int_{-\infty}^{t} J(t-t') \frac{d\sigma}{dt'} dt'$ and applying convolution theory.

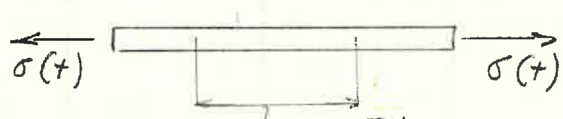
Note from myself: it can also be deduced immediately integrating by parts.

Relaxation Law

In the same way, the relaxation modulus $E(t)$ known from a test, we can write for any input $\epsilon = \epsilon(t)$

$$\sigma(t) = \epsilon_0 E(t) + \int_{0^+}^{t} E(t-t') \frac{d\epsilon}{dt'} dt'$$

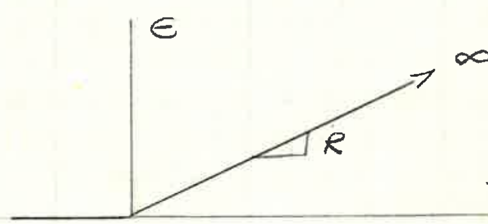
Example : constant strain-rate uniaxial test



R = constant rate

$d\epsilon/dt = R$

$$\sigma(t) = \cancel{\epsilon_0 E(t)}^{\,0} + R \int E(t-t') dt'$$

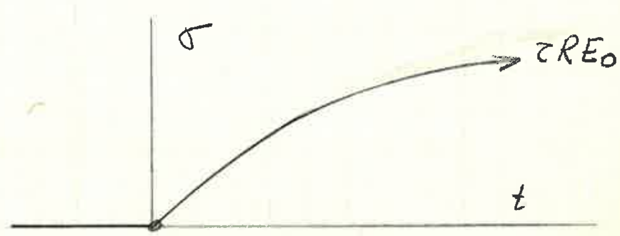


for a Maxwell solid

$$E(t) = E_0 \cdot \exp\left(-\frac{t}{\tau}\right)$$

$$E(t-t') = E_0 \exp\left(-\frac{t-t'}{\tau}\right)$$

$$\sigma(t) = \tau R E_0 \left[1 - \exp\left(-\frac{t}{\tau}\right)\right]$$



c) Mixed problem: combination of (a) + (b) often occurs (ex: cantilever beam loaded at the end)

1, 2, 3, 5 are INDEPENDENT of the material. Only through 4 (const. eq.) we introduce the kind of material.

Elastic Solids

1. Equilibrium equations $\tau_{ij,j} + \rho f_i = \rho \ddot{u}_i$

2. Strain-displac. " $\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$

3. Stress-strain laws $\tau_{ij} = \lambda \vartheta_1 \delta_{ij} + 2\mu \epsilon_{ij}$

we can substitute strains by displacements using (2):

$$\vartheta_1 = \epsilon_{ii} = u_{i,i}$$

3' $$\tau_{ij} = \lambda u_{i,i} \delta_{ij} + \mu (u_{i,j} + u_{j,i})$$

substituting 3' into the stress equilibrium equations, we get 1 in terms of displacements

$$u_{i,jj} = \frac{\partial^2 u_i}{\partial x_1^2} + \frac{\partial^2 u_i}{\partial x_2^2} + \frac{\partial^2 u_i}{\partial x_3^2} = \nabla^2 u_i$$

$$u_{j,ij} = \vartheta_{1,i}$$

$$\lambda \delta_{ij} \vartheta_{1,j} = \lambda \vartheta_{1,i}$$

$$\tau_{ij,j} = (\lambda + \mu)\vartheta_{1,i} + \mu \nabla^2 u_i$$

and

$$\mu \nabla^2 u_i + (\lambda + \mu)\vartheta_{1,i} + \rho f_i = \rho \ddot{u}_i \qquad i = 1, 2, 3$$

are the three Navier equilibrium equations, which reduce the 15 initial equations and 15 unknowns (6 stress, 6 strains and 3 displacements) to 3, at the cost of dealing with a 2nd order system. If $f_i = \ddot{u}_i = 0$ (elastostatic case — neither body forces nor acceleration) and the axis x, y, z:

$$\mu \nabla^2 u + (\lambda + \mu)\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right) = 0$$

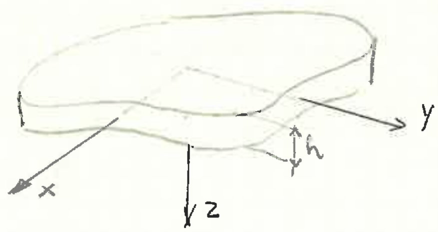
etc.

useful form when the displacements on S' are given

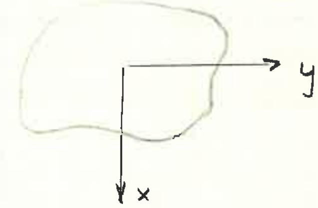
Plane problems

Stresses or strains depends only upon 2 dimensions

1. Generalized Plane Stress (Timoshenko calls this simply plane stress). Case of a Thin Plate or Slice



is replaced by its mid plane: $(z = 0)$



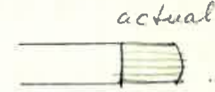
As the edges are free of stresses σ_z, τ_{xz}, τ_{yz}, and the plate is thin, we assume we can neglect them everywhere. Then

$$\sigma_z = \tau_{xz} = \tau_{yz} = 0$$

$$u, v, \sigma_x, \sigma_y \text{ and } \tau_{xy} = f(x, y)$$

Actually σ_x, σ_y, τ_{xy} are average values along the thickness

actual



The equilibrium equations are

$$\sigma_{x,x} + \tau_{xy,y} + \rho f_x = \rho \ddot{u}$$

$$\tau_{xy,x} + \sigma_{y,y} + \rho f_y = \rho \ddot{v}$$

and the stress-strains laws

$$\epsilon_x = \frac{\sigma_x}{E} - \frac{\nu \sigma_y}{E}, \quad \epsilon_y = \frac{\sigma_y}{E} - \frac{\nu \sigma_x}{E} \qquad \epsilon_z = -\frac{\nu}{E}(\sigma_x + \sigma_y)$$

2. Plane strain. Case of a slice cut from a long, prismatic cylinder, uniformly loaded along its axis z, $\epsilon_z = 0$ or constant

$$u, v = f(x, y) \rightarrow \tau_{xz} = 0 \quad \tau_{yz} = 0$$

$$\epsilon_x = \frac{\sigma_x}{E} - \frac{\nu \sigma_y}{E} \qquad \epsilon_y = \frac{\sigma_y}{E} - \frac{\nu \sigma_x}{E} \qquad \epsilon_z = 0 \therefore \sigma_z = -\nu(\sigma_x + \sigma_z)$$

the equilibrium equations are the same.

Plane elastostatic problems $(f_i = a_i = 0)$

Equilibrium equations

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0 \qquad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0$$

Strain stress equations:

$$\epsilon_x = \frac{\sigma_x}{E} - \frac{\nu\sigma_y}{E} \qquad \epsilon_y = \frac{\sigma_y}{E} - \frac{\nu\sigma_y}{E} \qquad \gamma_{xy} = \frac{1}{2\mu}\tau_{xy}$$

$$\tau_{xz} = 0 \qquad \tau_{yz} = 0$$

Plane stress $\quad \epsilon_z = -\frac{\nu}{E}(\sigma_x + \sigma_y) \qquad \sigma_z = 0$

" strain $\quad \epsilon_z = 0 \qquad \sigma_z = -\nu(\sigma_x + \sigma_y)$

Solution by stress function φ (Airy) Take

$$\sigma_x = \varphi_{,yy} \qquad \sigma_y = \varphi_{,xx} \qquad \tau_{xy} = -\varphi_{,xy}$$

equil. eq. are identically satisfied and the compatibility equation gives after substituting $\epsilon_x = \frac{1}{E}\varphi_{,yy} - \frac{\nu}{E}\varphi_{,xx}$ etc

$$\nabla^4 \varphi = 0$$

biharmonic equation. The sum $\sigma_x + \sigma_y$ satisfies the harmonic one $\nabla^2(\sigma_x + \sigma_y) = 0$.

References:

Sokolnikoff "Th of elasticity" rigorous treatment of 2-Dim elasticity.

For the application of complex variable based on the work of Muschelišvily school see Sokolnikoff or the Muschelišvily book "On the solution of some problems of Theory of elasticity."