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Moving least-square reproducing kernel method Part II: Fourier analysis

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Abstract

In Part I of this work, the moving least-square reproducing kernel (MLSRK) method is formulated and implemented. Based on its generic construction, an m-consistency structure is discovered and the convergence theorems are established.

In this part of the work, a systematic Fourier analysis is employed to evaluate and further establish the method. The preliminary Fourier analysis reveals that the MLSRK method is stable for sufficiently dense, non-degenerated particle distribution, in the sense that the kernel function family satisfies the Riesz bound. One of the novelties of the current approach is to treat the MLSRK method as a variant of the 'standard' finite element method and depart from there to make a connection with the multiresolution approximation. In the spirits of multiresolution analysis, we propose the following MLSRK transformation,

$$\mathcal{F}^{m,k}_{\varrho,h} u = \sum_{i=1}^{np} \langle u, \mathcal{K}_\varrho \rangle_i \breve{\mathcal{K}}^h_\varrho (x - x_i, x) w_i$$

The highlight of this paper is to embrace the MLSRK formulation with the notion of the controlled L_p-approximation. Based on its characterization, the Strang–Fix condition for example, a systematic procedure is proposed to design new window functions so they can enhance the computational performance of the MLSRK algorithm. The main effort here is to obtain a constant correction function in the interior region of a general domain, i.e. $C^h_\varrho = 1$. This can create a leap in the approximation order of the MLSRK algorithm significantly, if a highly smooth window function is embedded within the kernel. One consequence of this development is the synchronized convergence phenomenon—a unique convergence mechanism for the MLSRK method, i.e. by properly tuning the dilation parameter, the convergence rate of higher-order error norms will approach the same order convergence rate of the L_2 error norm—they are synchronized.

1. Introduction

As one of the favorite global partitions of unity, the moving least-square interpolation scheme is playing a major role in the increasingly popularized meshless methods (see [4,20,21]) for details.

Interestingly, its continuous counterpart—moving least-square reproducing kernel representation is, without exaggeration, reminiscent of the early 'spline plateau' related finite element subspace method, which was a milestone in the development of the mathematical theory of finite element methods. Once again, history repeats itself in an amazingly similar pattern. The new method is not just a rehash of the early theory. Instead of using the 'spline plateau' as the kernel function, the 'spline function' or 'hill function' is used here as the basic window function—a building block to generate a 'reproducing kernel',

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which possesses some excellent features such as reproducing m-order polynomials exactly in a random particle distribution, enforcing the essential boundary condition, as well as assuring global higher-order conforming property of the interpolation field, etc. This development brings new excitement and novel techniques into the state-of-the-art finite element technology. One such evidence is, as pointed out in Part I, that the current MLSRK method bears some resemblances with the critically hailed multiresolution analysis.

Naturally, for a scientific mind, it is not cynical to ask: is it a finite element method? is it a multiresolution approximation? or both? nor either? *'To be or not to be, that is the question.'* For application-minded engineers, their question might be rather straightforward and plain. What is the real 'touch' and novelty behind the theme, if it is not all hoopla. Due to the lack of understanding of how the algorithm works, we are unable to assess this meshless method precisely. Although a priori convergence estimates are given in Part I of this work [21], all the numerical experiments conducted so far suggest that the convergence results of the MLSRK are consistently better than the results obtained by using the same order finite element shape functions, and our theory fails to predict it.

As an indispensable analytical apparatus, Fourier analysis provides us with a unique instrument to transform a complex object into a simple, transparent image. Hence, in this part of the work, a systematic Fourier analysis is conducted to evaluate the MLSRK method in the spectral domain in order to gain some insight and understanding of its computational performance.

There are two departure points for our approach. One is the early abstract finite element formulation in terms of kernel functions, which was popular in mathematical finite element literature, e.g. Aubin [2], Babuška [3] and Strang and Fix [26–28]. Because of the lack of practical implementation, this formulation has remained as a symbolic abstraction overshadowed by a more engineering oriented triangularization and other piecewise polynomial based localization approaches. Of course, one could view the conventional piecewise continuous finite element shape function as a particular kernel function. Another source of inspiration is the state-of-the-art multiresolution approximation, which attracts a lot of attention these days. An enormous amount of literature has been published during a very short period on this subject. Here, our sources of reference are from three excellent monographs, Chui [5], Daubechies [7] and Meyer [22]. Readers can find many references in these books. Conceptually, multiresolution approximation is a perfect embodiment to implement the spectral–Galerkin method in solving partial differential equation numerically. As a matter of fact, numerous attempts have been initiated to explore the possibilities, and many claims have been announced, though some of them might be premature.

In this study, it is confirmed again that the MLSRK method is a multiresolution approximation, which is also related to the issue of numerical stability. In other words, being a numerical method, the numerical stability of MLSRK is assured in the sense that the discrete kernel function family consists of a frame, or even a Riesz basis, which is equivalent to saying that the condition number of the fundamental Gram matrix is bounded.

In the presentation, the MLSRK method is treated as a variant of the finite element method, and the analysis is focused on the comparison between its features in the physical space and those in the Fourier space. Section 2 mainly serves as an 'overture'; a handful of definitions are introduced. The main emphasis is placed on the discussion of the Strang–Fix condition [9,26], which plays a crucial role in our analysis. In Section 3, we develop a so-called finite element–moving least-square reproducing kernel (FE–MLSRK) formulation, which is parallel to the theory of general reproducing kernel formulation. Based on the concept of L_p-approximation, a Fourier transform version of the m-consistency condition is also established, which is particularly pertinent to the characterization of the MLSRK method in the Fourier space. A brief discussion on the relationship between numerical stability and multiresolution approximation is carried out in Section 4. In Section 5, we show how to systematically reconstruct window functions such that the correction function can be a constant in the interior domain, $C_\varrho^h = 1$, which substantially improves the approximation order of the reproducing kernel function. For instance, the kernel function can satisfy a higher-order Strang–Fix condition, which ultimately leads to a new convergence phenomenon—the synchronized convergence.

2. Preliminary

2.1. Fourier transform

There are many excellent monographs on the subject of the Fourier transform. However, as an engineering approach, our standard is moderate and application oriented. Therefore, our point of reference is mainly taken from Körner [18], Papoulis [23] and occasionally Stein and Weiss [31].

In most of this paper, we directly deal with the situations in $\mathbb{R}^n$, thus, the multi-index notation is used throughout the paper. A nuisance of the Fourier transform is that it has several definitions in the literature and each of them seems to have its own merit. Since the Fourier transform is only used as a tool in this work, we adopt the most common definition.

For $f(x) \in L_1(\mathbb{R}^n) \cap L_2(\mathbb{R}^n)$, the Fourier transform in $\mathbb{R}^n$ is defined as follows:

$$\hat{f}(\zeta) := \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x) \exp(-\mathrm{i}\zeta x)\, \mathrm{d}x = \int_{\mathbf{R}^n} f(x) \exp(-\mathrm{i}\zeta x)\, \mathrm{d}x \tag{1}$$

$$f(x) = \frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \hat{f}(\zeta) \exp(\mathrm{i}\zeta x)\, \mathrm{d}\zeta = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} \hat{f}(\zeta) \exp(\mathrm{i}\zeta x)\, \mathrm{d}\zeta \tag{2}$$

The notation

$$f(x) \Longleftrightarrow \hat{f}(\zeta) \tag{3}$$

is used to indicate that functions $f(x)$ and $\hat{f}(\zeta)$ are a Fourier transform pair related through Eqs. (1) and (2).

In the following, several useful results of the Fourier analysis are listed, so they can be conveniently employed at our disposal.

1. *Some elementary properties of the Fourier transform*

The following formulas can be found in [23], except (1e); in that case, the proof is outlined.

(1a) *Spatial shifting*

$$f(x - x_0) \Longleftrightarrow \hat{f}(\zeta) \exp(-\mathrm{i}x_0\zeta) \tag{4}$$

(1b) *Frequency shifting*

$$\exp(\mathrm{i}\zeta_0 x) f(x) \Longleftrightarrow \hat{f}(\zeta - \zeta_0) \tag{5}$$

(1c) *Spatial differentiation*

Assume $D_x^\alpha f(x) \in L_1(\mathbb{R}^n)$. Then,

$$D_x^\alpha f(x) \Longleftrightarrow (\mathrm{i}\zeta)^{|\alpha|} \hat{f}(\zeta) \tag{6}$$

(1d) *Frequency differentiation*

Assume $x^\alpha f(x) \in L_1(\mathbb{R}^n)$. Then,

$$(x)^\alpha f(x) \Longleftrightarrow (i)^{|\alpha|} D_\zeta^\alpha \hat{f}(\zeta) \tag{7}$$

(1e) *Scaling*

Let $f_\varrho(x) := 1/\varrho^n f(x/\varrho)$. One has

$$f_\varrho(x) \Longleftrightarrow \hat{f}(\varrho\zeta) \tag{8}$$

$$x^\alpha f_\varrho(x) \Longleftrightarrow (i)^{|\alpha|} \varrho^\alpha [D_\xi^\alpha \hat{f}(\xi)]|_{\xi=\varrho\zeta} \tag{9}$$

$$D_x^\alpha f_\varrho(x) \Longleftrightarrow (i)^{|\alpha|} \varrho^{-\alpha} [\xi^\alpha \hat{f}(\xi)]|_{\xi=\varrho\zeta} \tag{10}$$

PROOF.

(i) We first show (8). By definition (2),

$$\frac{1}{\varrho^n} f\left(\frac{x}{\varrho}\right) = \frac{1}{\varrho^n}\int_{\mathbf{R}^n} \hat{f}(\zeta)\exp\left(\mathrm{i}\zeta\frac{x}{\varrho}\right)\mathrm{d}\zeta$$
$$= \int_{\mathbf{R}^n} \hat{f}(\varrho\zeta)\exp(\mathrm{i}\zeta x)\,\mathrm{d}\zeta \tag{11}$$

(ii) Next we show (9). Note the fact that

$$x^\alpha f_\varrho(x) = \varrho^\alpha\left(\frac{1}{\varrho^n}\right)\left(\frac{x}{\varrho}\right)^\alpha f\left(\frac{x}{\varrho}\right) \tag{12}$$

By (8) and (7), one can derive (9) immediately.

(iii) Let $z = x/\varrho$. Consider

$$D_x^\alpha f_\varrho(x) = \varrho^{-\alpha} D_z^\alpha f(z) \tag{13}$$

and apply (8) and (6) to (13), to derive (10). □

(1f) *Convolution*

Suppose $f_1(x), f_2(x) \in L_1(\mathbb{R}^n) \cap L_2(\mathbb{R}^n)$ and

$$f_1(x) \Longleftrightarrow \hat{f}_1(\zeta); \qquad f_2(x) \Longleftrightarrow \hat{f}_2(\zeta) \tag{14}$$

then

$$\int_{\mathbf{R}^n} f_1(x-y) f_2(y)\,\mathrm{d}y \Longleftrightarrow \hat{f}_1(\zeta)\hat{f}_2(\zeta) \tag{15}$$

2. *The general moment theorem*

The connection between derivatives of the Fourier transform of a function $\phi(\zeta)$ and its moments is well-known and can be stated as follows:

THEOREM 2.1 ([23]). $\forall x^\alpha\phi(x) \in L_1(\mathbb{R}^n)$, let

$$m_\alpha := \int_{\mathbf{R}^n} z^\alpha \phi(z)\,\mathrm{d}z \tag{16}$$

Then,

$$m_\alpha = (i)^{|\alpha|} D_\zeta^\alpha \hat{\phi}(0) \tag{17}$$

where the n-tuple α is

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \tag{18}$$

The proof of the theorem is omitted here; one can also find it in [23].

The above conventional moment theorem can be extended to moments of derivatives of a function ϕ, which we referred to as the *'general moment theorem'*.

THEOREM 2.2 (The general moment theorem). If $x^\alpha\phi(x) \in L_1(\mathbb{R}^n)$, let

$$m_\alpha^\beta := \int_{\mathbf{R}^n} z^\alpha D_z^\beta \phi(z)\,\mathrm{d}z \tag{19}$$

where $D_z^\beta := \partial_{z_1}^{\beta_1}\partial_{z_2}^{\beta_2}\cdots\partial_{z_n}^{\beta_n}$.

One has

$$m_\alpha^\beta = (i)^{|\alpha+\beta|}\frac{\alpha!}{(\alpha-\beta)!} D_\zeta^{\alpha-\beta}\hat{\phi}(0)\langle\alpha-\beta\rangle \tag{20}$$

where

$$\langle \alpha - \beta \rangle := \begin{cases} 1, & |\alpha| \geqslant |\beta| \\ 0, & |\alpha| < |\beta| \end{cases} \tag{21}$$

PROOF. By definition

$$\phi(x) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} \hat{\phi}(\zeta) \exp(\mathrm{i}\zeta x)\, \mathrm{d}\zeta \tag{22}$$

Consequently,

$$D_x^{\beta} \phi(x) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} (\mathrm{i}\zeta)^{\beta} \hat{\phi}(\zeta) \exp(\mathrm{i}\zeta x)\, \mathrm{d}\zeta \tag{23}$$

$$\Rightarrow (\mathrm{i}\zeta)^{\beta} \hat{\phi}(\zeta) = \int_{\mathbf{R}^n} D_x^{\beta} \phi(x) \exp(-\mathrm{i}\zeta x)\, \mathrm{d}x \tag{24}$$

Substituting $\hat{f}(\zeta) = (\mathrm{i}\zeta)^{\beta} \hat{\phi}(\zeta)$ into (7) yields

$$(i)^{|\alpha+\beta|} D_{\zeta}^{\alpha} [\zeta^{\beta} \hat{\phi}(\zeta)] = \int_{\mathbf{R}^n} x^{\alpha} D_x^{\beta} \phi(x) \exp(-\mathrm{i}\zeta x)\, \mathrm{d}x \tag{25}$$

Finally, we have

$$\int_{\mathbf{R}^n} x^{\alpha} D_x^{\beta} \phi(x) \exp(-\mathrm{i}\zeta x)\, \mathrm{d}x = (i)^{|\alpha+\beta|} \sum_{\gamma \leqslant \alpha} \binom{\alpha}{\gamma} D_{\zeta}^{\alpha-\gamma} [\hat{\phi}(\zeta)] D_{\zeta}^{\gamma} [\zeta^{\beta}] \tag{26}$$

$$= (i)^{|\alpha+\beta|} \sum_{\gamma \leqslant \alpha} \binom{\alpha}{\gamma} D_{\zeta}^{\alpha-\gamma} \hat{\phi}(\zeta) \frac{\beta!}{(\beta-\gamma)!} \zeta^{\beta-\gamma} \langle \beta - \gamma \rangle \tag{27}$$

Let $\zeta = 0$. Eq. (27) immediately yields

$$m_{\alpha}^{\beta} = (i)^{|\alpha+\beta|} \frac{\alpha!}{(\alpha-\beta)!} D_{\zeta}^{\alpha-\beta} \hat{\phi}(0) \langle \alpha - \beta \rangle \tag{28}$$

Note that Eqs. (25), (26) and (27) will be repeatedly used. □

3. *Bessel's inequality ([8,18])*

(1) If $\{e_j\}$ is an infinite sequence of orthonormal elements and $f \in C(\mathbb{R}^n)$ is continuous, then

$$\|f\|^2_{L_2(\mathbf{R}^n)} \geqslant \sum_{j \in \mathbf{Z}^n} |(f, e_j)|^2 \tag{29}$$

(2) Let

$$\boldsymbol{T} := \mathbb{R}/2\pi\mathbf{Z} \tag{30}$$

If $f : \boldsymbol{T} \to \mathbb{C}$ is continuous, then

$$\frac{1}{(2\pi)^n} \int_{\boldsymbol{T}^n} |f(t)|^2\, \mathrm{d}t \geqslant \sum_{j \in \mathbf{Z}^n} |\hat{f}(j)|^2 \tag{31}$$

4. *Poisson's summation formula ([5])*

Assume:

(i) $f \in L_1(\mathbb{R}^n)$;
(ii) $\sum_{j \in \mathbf{Z}^n} f(x + 2\pi j)$ converges everywhere to a continuous function;
(iii) $1/(2\pi)^n \sum_{j \in \mathbf{Z}^n} \hat{f}(j) \exp(\mathrm{i}jx)$ converges everywhere;

then

$$\sum_{j\in \mathbf{Z}^n} f(x+2\pi j) = \frac{1}{(2\pi)^n}\sum_{j\in \mathbf{Z}^n}\hat{f}(j)\exp(\mathrm{i}jx), \quad x\in \mathbb{R}^n \tag{32}$$

in particular, $x=0$

$$\sum_{j\in \mathbf{Z}^n} f(2\pi j) = \frac{1}{(2\pi)^n}\sum_{j\in \mathbf{Z}^n}\hat{f}(j) \tag{33}$$

If the x-axis is scaled by a factor a, then, Eq. (32) may be rewritten as

$$\sum_{j\in \mathbf{Z}^n} f(x+2\pi aj) = \frac{1}{(2\pi a)^n}\sum_{j\in \mathbf{Z}^n}\hat{f}(j/a)\exp\left(\mathrm{i}\frac{jx}{a}\right) \tag{34}$$

where

$$j/a := (j_1/a_1, j_2/a_2, \dots, j_n/a_n) \tag{35}$$

$$aj := (a_1j_1, a_2j_2, \dots, a_nj_n) \tag{36}$$

Let $x=-x$, $a=h/2\pi$ and $x_j=hj$. One has

$$\sum_{j\in \mathbf{Z}^n} f(x_j-x) = \frac{1}{h^n}\sum_{j\in \mathbf{Z}^n}\hat{f}\left(\frac{2\pi j}{h}\right)\exp\left(-\mathrm{i}\frac{2\pi jx}{h}\right) \tag{37}$$

$$x=0 \ \Rightarrow\ \sum_{j\in \mathbf{Z}^n} f(hj) = \frac{1}{h^n}\sum_{j\in \mathbf{Z}^n}\hat{f}\left(\frac{2\pi j}{h}\right) \tag{38}$$

Vice versa, one can have the Poisson's summation formula expressed for the sampled data [23]

$$\sum_{j\in \mathbf{Z}^n}\hat{f}\left(\zeta+\frac{2\pi j}{h}\right) = h^n\sum_{j\in \mathbf{Z}^n} f(hj)\exp(-\mathrm{i}hj\zeta) \tag{39}$$

2.2. Strang–Fix condition

The Strang–Fix condition is essentially a condition to categorize functions based on their regularity [26,9,17]. Before discussing the Strang–Fix condition, it would be expedient to make a few definitions. The Schwartz class $\mathcal{S}_r$ is introduced as follows [29].

DEFINITION 2.1. The class of r-order regular functions, $\mathcal{S}_r$, is defined as

$$\mathcal{S}_r := \left\{\phi \;\middle|\; |D^\alpha\phi| \leqslant \frac{c_{\alpha m}}{(1+|x|)^m}\ ,\ |\alpha|\leqslant r,\ \forall m\in\mathbb{N}\right\} \tag{40}$$

We denote $\mathcal{B}$ as a subset of $\mathcal{S}_r$,

$$\mathcal{B} := \left\{\phi \;\middle|\; \phi\in L_1(\mathbb{R}^n), |\phi|\leqslant \frac{c}{(1+|x|)^k},\ k>1\right\} \tag{41}$$

Now, we are in a position to state the Strang–Fix condition.

DEFINITION 2.2 (Strang–Fix Condition). Assume $x^\alpha\phi(x)\in L_1(\mathbb{R}^n)$, the function $\phi(x)$ is said to satisfy the pth order Strang–Fix condition, if the following equalities hold

$$\hat{\phi}(0)=1 \tag{42}$$

$$D^\alpha_\zeta\hat{\phi}(\zeta)\,|_{2\pi j}=0\ ,\quad \forall j\in\mathbf{Z}^n\backslash\{0\} \tag{43}$$

where $|\alpha|\leqslant p$.

The condition (43) can be slightly generalized as

$$D_\zeta^\alpha \hat{\phi}(\zeta) \mid_{\frac{2\pi j}{h}} = 0 \ , \quad \forall j \in \mathbf{Z}^n \backslash \{0\} \tag{44}$$

where

$$j/h := (j_1/h_1, j_2/h_2, \ldots, j_n/h_n) \tag{45}$$

The Strang–Fix condition possesses some useful properties. The following statement about the Strang–Fix condition is a generalized version of a proposition made by Daubechies et al. [10].

PROPOSITION 2.1. The set of functions that satisfies the pth order semi-Strang Fix condition,

$$\mathcal{SF}_h^{(p)} := \left\{ \phi \mid x^\alpha \phi(x) \in \mathcal{B}; D_\zeta^\alpha \hat{\phi}(\zeta) \mid_{\zeta = \frac{2\pi j}{h}} = 0, \ \forall j \in \mathbf{Z}^n \backslash \{0\}, |\alpha| \leqslant p \right\} \tag{46}$$

have the following properties

(i) $\mathcal{SF}_h^{(p)}$ *is translation invariance:*

$$\phi(x) \in \mathcal{SF}_h^{(p)} \Rightarrow \phi(x+t) \in \mathcal{SF}_h^{(p)}; \quad t \in \mathbb{R}^n \tag{47}$$

(ii) $\mathcal{SF}_h^{(p)}$ *is scaling invariance in a sense:*

$$\phi(x) \in \mathcal{SF}_h^{(p)}, \quad t > 0 \Rightarrow \phi(tx) \in \mathcal{SF}_{th}^{(p)}; \quad t \in \mathbb{R}^n \tag{48}$$

(iii) $\mathcal{SF}_h^{(p)}$ *is differentiation invariance:*

$$\phi \in \mathcal{S}_r \quad \text{and} \quad \phi(x) \in \mathcal{SF}_h^{(p)} \Rightarrow D_\zeta^\alpha \phi(x) \in \mathcal{SF}_h^{(p)}; \tag{49}$$

(iv) $\mathcal{SF}_h^{(p)}$ *is an ideal under convolution, i.e.*

$$\phi(x), \psi(x) \in \mathcal{SF}_h^{(p)} \Rightarrow \phi * \psi \in \mathcal{SF}_h^{(p)} \tag{50}$$

(v) $\mathcal{SF}_h^{(p)}$ *is an ideal under* $(p-1)$*-order weighted convolution, i.e.*

$$|\mu + \nu| \leqslant p - 1, \quad \phi(x), \psi(x) \in \mathcal{SF}_h^{(p)} \Rightarrow \left(x^\mu \phi(x)\right) * \left(x^\nu \psi(x)\right) \in \mathcal{SF}_h^{(p)} \tag{51}$$

(vi) If $\phi(x) \in \mathcal{SF}_h^{(p)}$*, the* α*th moment of* ϕ *can be written as*

$$\int_{\mathbf{R}^n} x^\alpha \phi(x)\, \mathrm{d}x = \sum_{x_j \in \mathbf{H}^n} (x_j - x)^\alpha \phi(x_j - x) h^n, \quad |\alpha| \leqslant p \tag{52}$$

(vii) If $\phi(x) \in \mathcal{SF}_h^{(p)}$*, the* α*th moment of* $D_x^\beta \phi$ *can be written as*

$$\int_{\mathbf{R}^n} x^\alpha D_x^\beta \phi(x)\, \mathrm{d}x = \sum_{x_j \in \mathbf{H}^n} (x_j - x)^\alpha D_x^\beta \phi(x_j - x) h^n, \quad |\alpha|, |\beta| \leqslant p \tag{53}$$

where $\mathbb{H}^n = h\mathbf{Z}^n$*. Note that here the differential operator* D_x^β *is acting on the argument* $(x_j - x)$ *rather than the argument* x*.*

PROOF. Properties (i)–(iii) are obvious. Property (iv) is a special case of (v); and property (vi) is a special case of property (vii). Hence, it suffices to show that (v) and (vii) are true.

(1) Property (v);
One has to show that

$$D_\zeta^\alpha (D_\zeta^\mu \hat{\phi}(\zeta) D_\zeta^\nu \hat{\phi}(\zeta))|_{\zeta = \frac{2\pi j}{h}} = 0, \quad \forall |\mu + \nu| \leqslant p - 1, j \in \mathbf{Z}^n \backslash \{0\} \tag{54}$$

We proceed by showing the contrary. By the Leibnitz rule, one can have

$$D_\zeta^\alpha(D_\zeta^\mu\hat{\phi}(\zeta)D_\zeta^\nu\hat{\psi}(\zeta)) = \binom{\alpha}{0}D_\zeta^{\alpha+\mu}\hat{\phi}(\zeta)D_\zeta^\nu\hat{\psi}(\zeta) + \binom{\alpha}{1}D_\zeta^{\alpha-1+\mu}\hat{\phi}(\zeta)D_\zeta^{1+\nu}\hat{\psi}(\zeta)$$
$$+\cdots+\binom{\alpha}{\beta}D_\zeta^{\alpha-\beta+\mu}\hat{\phi}(\zeta)D_\zeta^{\beta+\nu}\hat{\psi}(\zeta)$$
$$+\cdots+\binom{\alpha}{\alpha}D_\zeta^{\mu}\hat{\phi}(\zeta)D_\zeta^{\alpha+\nu}\hat{\psi}(\zeta) \tag{55}$$

Suppose the condition (54) is not true. Then, $\exists\beta$ such that $|\beta| \leqslant |\alpha|$ and

$$|\alpha-\beta+\mu| \geqslant p \quad \text{and} \quad |\beta+\nu| \geqslant p \tag{56}$$

Combining the two yields

$$|\alpha+\mu+\nu| \geqslant 2p \Rightarrow |\mu+\nu| \geqslant p \tag{57}$$

This violates the assumption $|\mu+\nu| \leqslant p-1$. Thereby, property (v) holds.

(2) Property (vii);
Let

$$G(y,x) := (y-x)^\alpha D_y^\beta\phi(y-x,x)h^n \tag{58}$$

The Fourier transform of $G(y,x)$ will be (for argument y),

$$\hat{G}(\zeta,x) = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} h^n(y-x)^\alpha D_y^\beta\phi(y-x,x)\exp[-\mathrm{i}\zeta y]\,\mathrm{d}y$$
$$= \exp[-\mathrm{i}\zeta x]h^n\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} z^\alpha D_z^\beta\phi(z,x)\exp[-\mathrm{i}\zeta z]\,\mathrm{d}z$$
$$= (i)^{|\alpha+\beta|}\exp[-\mathrm{i}\zeta x]h^n D_\zeta^\alpha[\zeta^\beta\hat{\phi}(\zeta,x)] \tag{59}$$

Let $x_j = hj$. The Poisson summation formula (38) yields

$$\sum_{x_j\in\mathbf{H}^n}(x_j-x)^\alpha D_x^\beta\phi(x_j-x)h^n = \sum_{j\in\mathbf{Z}^n}G(jh,x)$$
$$= \frac{1}{h^n}\sum_{j\in\mathbf{Z}^n}\hat{G}\left(\frac{2\pi j}{h},x\right) \Leftarrow \text{by (38)}$$
$$= (i)^{|\alpha+\beta|}\sum_{j\in\mathbf{Z}^n}D_\zeta^\alpha\left[(\zeta)^\beta\hat{\phi}(\zeta)\right]\Big|_{\zeta=\frac{2\pi j}{h}} \tag{60}$$

By considering the fact that $\phi(x)\in\mathcal{SF}_h^{(p)}$, Eq. (60) can be further simplified as follows:

$$\sum_{x_j\in\mathbf{H}^n}(x_j-x)^\alpha D_x^\beta\phi(x_j-x)h^n = (i)^{|\alpha+\beta|}\sum_{j\in\mathbf{Z}^n}\exp\left[-\mathrm{i}\frac{2\pi jx}{h}\right]\left\{\binom{\alpha}{0}(D_\zeta^\alpha\hat{\phi}(2\pi j/h))\left(\frac{2\pi j}{h}\right)^\beta\right.$$
$$+\binom{\alpha}{1}(D_\zeta^{\alpha-1}\hat{\phi}(2\pi j/h))\beta\left(\frac{2\pi j}{h}\right)^{\beta-1}+\cdots$$
$$\left.+\cdots\binom{\alpha}{\beta}(D_\zeta^{\alpha-\beta}\hat{\phi}(2\pi j/h))(\beta!)\right\}$$
$$= (i)^{|\alpha+\beta|}\frac{\alpha!}{(\alpha-\beta)!}[D_\zeta^{\alpha-\beta}\hat{\phi}(0)]\langle\alpha-\beta\rangle \tag{61}$$

On the other hand, by the general moment theorem (20),

$$m_\alpha^\beta := \int_{\mathbf{R}^n}x^\alpha D_x^\beta\phi(x)\,\mathrm{d}x = (i)^{|\alpha+\beta|}\frac{\alpha!}{(\alpha-\beta)!}D_\zeta^{\alpha-\beta}\hat{\phi}(0)\langle\alpha-\beta\rangle \tag{62}$$

Comparing (61) and (62) yields property (vii), i.e. Eq. (53). □

It should be noted that today the condition that bears the names of Strang and Fix has become a very useful categorization of functions in the field of harmonic analysis as well as approximation theory [14,16,17]. As Gilbert Strang himself commented, '*It is true that this has now become the Strang–Fix condition, much more useful than I ever expected!*' [30]

3. Mathematical analysis

3.1. The finite element representation

At this point, it is most likely considered self-indulgent to juxtapose the meshless method with the finite element method. In this paper, however, we are attempting to 'fit' the moving least-square reproducing kernel method into a form of the standard finite element formulation. This 'standard' finite element formulation is mainly due to Aubin [2], and was considered as a paradigm of the charming, elegant and stylish work of French school. It might sound a little bit antiqueted today; nonetheless, there is no doubt that the MLSRK method inherits its bourgeois ancestor's legacy. Indeed, the MLSRK method and the finite element method virtually share the same origin.

In Aubin's original presentation, there are only two basic operations: *restriction* and *prolongation*. We add a third operation—*reproducing operation*, to unify the whole formulation.

We start by defining the integral reproducing kernel representation.

DEFINITION 3.1 (Reproducing operation). The mth order reproducing operation is defined as

$$\mathcal{R}^m : C(\mathbb{R}^n) \mapsto C^m(\mathbb{R}^n) \tag{63}$$

$$\mathcal{R}^m_\varrho u := \int_{\mathbf{R}^n} \mathcal{K}_\varrho(y-x)u(y)\,\mathrm{d}y$$

where

$$\mathcal{K}_\varrho(x) := \frac{1}{\varrho^n}\mathcal{K}\left(\frac{x}{\varrho}\right) \tag{64}$$

In this paper, the kernel function is specifically referred to as the MLSRK function ([21] or Appendix A)

$$\mathcal{K}_\varrho(y-x) := \frac{1}{\varrho^n}\boldsymbol{P}(0)\boldsymbol{M}_\varrho^{-1}\boldsymbol{P}^t\left(\frac{y-x}{\varrho}\right)\phi\left(\frac{y-x}{\varrho}\right) \tag{65}$$

A discrete version of the moving least-square reproducing kernel approximation is proposed as [21]

$$\mathcal{R}^m_{\varrho,h} : \mathbb{H}^n \mapsto C^m(\mathbb{R}^n) \tag{66}$$

$$\mathcal{R}^m_{\varrho,h}u := \sum_{j\in\boldsymbol{Z}} \mathcal{K}^h_\varrho(x_j - x, x)u^h_j w_j \tag{67}$$

where the discrete kernel $\mathcal{K}^h_\varrho$ is defined as follows:

$$\mathcal{K}^h_\varrho(x_j - x, x) := \frac{1}{\varrho^n}\boldsymbol{P}(0)\left(\boldsymbol{M}^h_\varrho(x)\right)^{-1}\boldsymbol{P}^t\left(\frac{x_j-x}{\varrho}\right)\phi\left(\frac{x_j-x}{\varrho}\right) \tag{68}$$

in which

$$\boldsymbol{M}^h_\varrho(x) := \{m^h_{ij}(x)\} \tag{69}$$

and

$$m^h_{ij}(x) = \frac{1}{\varrho^n}\sum_{j\in\boldsymbol{Z}^n}\left(\frac{x_j-x}{\varrho}\right)^{(\alpha_{i-1}+\alpha_{j-1})}\phi\left(\frac{x_j-x}{\varrho}\right) \tag{70}$$

$$\boldsymbol{P}\left(\frac{x_j-x}{\varrho}\right) := \left\{1, \left(\frac{x_j-x}{\varrho}\right)^{\alpha_1}, \left(\frac{x_j-x}{\varrho}\right)^{\alpha_2}, \ldots, \left(\frac{x_j-x}{\varrho}\right)^{\alpha_{\ell-1}}\right\} \tag{71}$$

The weight w_j can be assigned based on quadrature rules, trapezoidal rule for instance, such that $\forall\, \Omega \subset \mathbb{R}^n$,

$$\sum_{x_j \in \Omega} w_j = \text{meas}(\Omega) \tag{72}$$

Or, simply let

$$w_j = 1, \quad \forall j \in \mathbf{Z}^n \tag{73}$$

In Part I, we have shown that both the continuous moving least-square reproducing kernel approximation and the discrete moving least-square reproducing kernel approximation are the mth order L_p-approximation. Both of them satisfy a so-called *m-order consistency* condition. For convenience, both the continuous version as well as the discrete version of m-consistency conditions are listed as follows:

$$\begin{cases} \displaystyle\int_{\mathbf{R}^n} (y-x)^\alpha \mathcal{K}_\varrho(y-x)\,\mathrm{d}y = \delta_{\alpha 0}, & |\alpha| \leqslant m \\ \displaystyle\int_{\mathbf{R}^n} (y-x)^\alpha D_x^\beta \mathcal{K}_\varrho(x-y)\,\mathrm{d}y = \alpha!\,\delta_{\alpha\beta}, & |\alpha|, |\beta| \leqslant m \end{cases} \tag{74}$$

Or, let $\check{\mathcal{K}}(x) := \mathcal{K}(-x)$

$$\begin{cases} \displaystyle\int_{\mathbf{R}^n} y^\alpha \check{\mathcal{K}}_\varrho(x-y)\,\mathrm{d}y = x^\alpha, & |\alpha| \leqslant m \\ \displaystyle\int_{\mathbf{R}^n} y^\alpha D_x^\beta \check{\mathcal{K}}_\varrho(x-y)\,\mathrm{d}y = \frac{\alpha!}{(\alpha-\beta)!}\,x^{\alpha-\beta}, & |\alpha|, |\beta| \leqslant m \end{cases} \tag{75}$$

And

$$\begin{cases} \displaystyle\sum_{j \in \mathbf{Z}^n} (x_j - x)^\alpha \mathcal{K}_\varrho^h(x_j - x, x) w_j = \delta_{\alpha 0}, & |\alpha| \leqslant m \\ \displaystyle\sum_{j \in \mathbf{Z}^n} (x_j - x)^\alpha D_x^\beta \mathcal{K}_\varrho^h(x_j - x, x) w_j = \alpha!\,\delta_{\alpha\beta}, & |\alpha|, |\beta| \leqslant m \end{cases} \tag{76}$$

Or, let $\check{\mathcal{K}}_\varrho^h(y,x) = \mathcal{K}_\varrho^h(-y,x)$.

$$\begin{cases} \displaystyle\sum_{j \in \mathbf{Z}^n} x_j^\alpha \check{\mathcal{K}}_\varrho^h(x - x_j, x) w_j = x^\alpha, & |\alpha| \leqslant m \\ \displaystyle\sum_{j \in \mathbf{Z}^n} x_j^\alpha D_x^\beta \check{\mathcal{K}}_\varrho^h(x - x_j, x) w_j = \frac{\alpha!}{(\alpha-\beta)!}\,x^{\alpha-\beta}, & |\alpha|, |\beta| \leqslant m \end{cases} \tag{77}$$

For more detailed information, readers may consult Part I of this work [21].

DEFINITION 3.2 (Restriction). Let $\mathbb{H}^n = h\mathbf{Z}^n, h := (h_1, h_2, \ldots, h_n)$. The restriction operation is defined as a mapping

$$r_h : C(\mathbb{R}^n) \mapsto V_h(\mathbb{H}^n) \tag{78}$$

$$r_h u = \int_{\mathbf{R}^n} \frac{1}{\varrho^n} \lambda\left(\frac{y}{\varrho} - j/a\right) u(y)\,\mathrm{d}y \tag{79}$$

where $a := \varrho/h$, and

$$\int_{\mathbf{R}^n} \lambda(y)\,\mathrm{d}y = 1 \tag{80}$$

In the following, several restriction operations, or choices of restriction kernel functions λ, are illustrated as the examples.

EXAMPLE 3.1 (The n-dimensional δ-comb). Let

$$\lambda_0(y - x_j) = \delta(y - x_j) \tag{81}$$

Choosing $a = 1$, we have

$$\begin{aligned} r_h^0 u &= \int_{\mathbf{R}^n} \frac{1}{h^n} \lambda_0 \left(\frac{y}{h} - j\right) u(y)\,\mathrm{d}y \\ &= \int_{\mathbf{R}^n} \frac{1}{h^n} \delta \left(\frac{y}{h} - j\right) u(y)\,\mathrm{d}y = u_j^h \end{aligned} \tag{82}$$

EXAMPLE 3.2 (Shannon sampling). Let

$$\lambda_s(y - x) := \frac{\sin \sigma(y - x)}{\pi(y - x)} := \prod_{i=1}^{n} \frac{\sin\{\sigma(y^i - x^i)\}}{\pi(y^i - x^i)} \tag{83}$$

Then, one can define the restriction operation r_h^s as

$$r_h^s u := \frac{1}{h^n} \int_{\mathbf{R}^n} \lambda_s \left(\frac{y}{h} - j\right) u(y)\,\mathrm{d}y = \frac{1}{h^n} \int_{\mathbf{R}^n} \frac{\sin \sigma(\frac{y}{h} - j)}{\pi(\frac{y}{h} - j)} u(y)\,\mathrm{d}y \tag{84}$$

EXAMPLE 3.3 (MLSRK transformation). Let

$$\lambda_k(y - x) := \mathcal{K}_\varrho(y - x) \tag{85}$$

Then

$$r_h^k u = \int_{\mathbf{R}^n} \frac{1}{\varrho^n} \lambda_k \left(\frac{y}{h} - j/a\right) u(y)\,\mathrm{d}y = \int_{\mathbf{R}^n} \frac{1}{\varrho^n} \mathcal{K} \left(\frac{y - x_j}{\varrho}\right) u(y)\,\mathrm{d}y \tag{86}$$

$$= \int_{\mathbf{R}^n} \frac{1}{\varrho^n} \boldsymbol{P}(0) \boldsymbol{M}_\varrho^{-1} \boldsymbol{P}^t \left(\frac{y - x_j}{\varrho}\right) \phi \left(\frac{y - x_j}{\varrho}\right) u(y)\,\mathrm{d}y \tag{87}$$

Note that in this case $\varrho \neq h$, i.e. $a \neq 1$.

DEFINITION 3.3 (Prolongation). The prolongation operation is defined as the mapping

$$p_\varrho^m u^h : V^h(\mathbb{H}^n) \mapsto C^m(\mathbb{R}^n) \tag{88}$$

$$p_\varrho^m u^h = \sum_{j \in \mathbf{Z}^n} \check{\mathcal{K}}_\varrho^h(x - x_j, x) u_j^h w_j \tag{89}$$

where

$$\begin{aligned} \check{\mathcal{K}}_\varrho^h(x - x_j, x) &:= \mathcal{K}_\varrho{}^h(x_j - x, x) \\ &= \frac{1}{\varrho^n} \boldsymbol{P}(0) \boldsymbol{M}_\varrho^{-1}(x) \boldsymbol{P}^t \left(\frac{x_j - x}{\varrho}\right) \phi \left(\frac{x_j - x}{\varrho}\right) \end{aligned} \tag{90}$$

It should be noted that $\mathcal{K}_\varrho^h(y, x) = \mathcal{K}_\varrho^h(-y, x)$, $\mathcal{K}_\varrho(x) = \mathcal{K}_\varrho(-x)$, if only the window function is symmetric, i.e. $\phi(x) = \phi(-x)$. The distinction will be explicitly emphasized when a particular situation is encountered.

To this end, we are in a position to define a so-called finite element–moving least-square reproducing kernel (FE–MLSRK) approximation.

DEFINITION 3.4 (Finite Element–Moving Least Square Reproducing Kernel Method). The following composite mapping of restriction operation and prolongation operation is defined as finite element–moving least-square reproducing kernel approximation, if we choose the prolongation kernel function as a product of MLSRK function with an appropriate weight.

$$\mathcal{F}^{m,q}_{\varrho,h} u := (p^m_\varrho \circ r^q_h)\, u : C(\mathbb{R}^n) \mapsto C^m(\mathbb{R}^n) \tag{91}$$

$$\mathcal{F}^{m,q}_{\varrho,h} u = \sum_{j \in \mathbf{Z}^n} \left(\int_{\mathbf{R}^n} \frac{1}{\varrho^n} \lambda_q \left(\frac{y}{\varrho} - j/a \right) u(y)\, \mathrm{d}y \right) \check{\mathcal{K}}^h_\varrho(x - x_j, x) w_j \tag{92}$$

Eq. (92) can be rewritten in a form of inner product

$$\mathcal{F}^{m,q}_{\varrho,h} u = \sum_{j \in \mathbf{Z}^n} \langle u, \lambda_q \rangle_j \check{\mathcal{K}}^h_\varrho(x - x_j, x) w_j \tag{93}$$

where

$$\langle u, \lambda_q \rangle_j := \frac{1}{\varrho^n} \int_{\mathbf{R}^n} \lambda_q \left(\frac{y}{\varrho} - j/a \right) u(y)\, \mathrm{d}y \tag{94}$$

We call this the finite element–moving least-square reproducing kernel (FE–MLSRK) representation.

If one chooses $q = 0$, Eq. (93) recovers the original MLSRK formula, i.e.

$$\begin{aligned}\mathcal{F}^{m,0}_{\varrho,h} u &= (p^m_\varrho \circ r^0_h) u = \sum_{j \in \mathbf{Z}} \check{\mathcal{K}}^h_\varrho(x - x_j, x) u^h_j w_j \\ &= \sum_{j \in \mathbf{Z}} \mathcal{K}^h_\varrho(x_j - x, x) u^h_j w_j = \mathcal{R}^m_{\varrho,h} u\end{aligned} \tag{95}$$

If one chooses $q = k$, Eq. (93) represents a new MLSRK formulation—a genuine reproducing kernel formula.

$$\mathcal{F}^{m,k}_{\varrho,h} u = \sum_{j \in \mathbf{Z}^n} \left(\int_{\mathbf{R}^n} \mathcal{K}_\varrho(y - x_j) u(y)\, \mathrm{d}y \right) \check{\mathcal{K}}^h_\varrho(x - x_j, x_j) w_j \tag{96}$$

$$= \sum_{j \in \mathbf{Z}^n} \langle u, \mathcal{K}_\varrho \rangle_j \check{\mathcal{K}}^h_\varrho(x - x_j, x) w_j \tag{97}$$

In practical implementation, the following double series summation is used

$$\tilde{\mathcal{F}}^{m,k}_{\varrho,h} u = \sum_{j \in \mathbf{Z}^n} \left(\sum_{i \in \mathbf{Z}^n} \mathcal{K}^h_\varrho(x_i - x_j, x_j) u_i w_i \right) \check{\mathcal{K}}^h_\varrho(x - x_j, x) w_j \tag{98}$$

The inner summation can be interpreted as the discrete MLSRK transform, and the outer summation can be considered as the inverse MLSRK transform.

One can easily extend the formula (98) into the cases of finite region, as well as random particle distribution, i.e.

$$\tilde{\mathcal{F}}^{m,k}_{\varrho,h} u = \sum_{j=1}^{np} \left(\sum_{i=1}^{np} \mathcal{K}^h_\varrho(x_i - x_j, x_j) u_i w_i \right) \check{\mathcal{K}}^h_\varrho(x - x_j, x) w_j \tag{99}$$

where $x_i \neq hi$ and $x_j \neq hj$.

REMARK 3.1. In Aubin's finite element formulation [2], in order to obtain a L_p-approximation, the prolongation kernel function must satisfy the following *m-convergence* condition

$$\sum_{k \in \mathbf{Z}^n} \frac{k^j}{j!} \mu(x - k) = \sum_{0 \leqslant k \leqslant j} b^k \frac{x^{j-k}}{(j-k)!}, \quad 0 \leqslant j \leqslant m \tag{100}$$

where b^k are constants. Furthermore, the restriction kernel function and the prolongation kernel function have to be compatible with each other, i.e.

$$\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} \mu(x)\lambda(y)(y-x)^\alpha \,\mathrm{d}x\,\mathrm{d}y = \begin{cases} 1\,, & \alpha = 0 \\ 0\,, & 0 < |\alpha| \leqslant m \end{cases} \tag{101}$$

In our FE–MLSRK formulation, the prolongation kernel function is specifically assigned as the moving least-square reproducing kernel function,

$$\mu(x) = \check{\mathcal{K}}(x), \tag{102}$$

that satisfies the m-consistency condition (76)—a particular form of m-convergence condition (100). It should be noted that the m-consistency condition (76) is a much stronger condition than m-convergence condition. Furthermore, by using Eq. (74), one may find that

$$\int_{\mathbf{R}^n} \check{\mathcal{K}}(x)(y-x)^\alpha \,\mathrm{d}x = y^\alpha \tag{103}$$

To meet the compatible condition (101), this requires that the restriction kernel function satisfies the condition

$$\int_{\mathbf{R}^n} \lambda(y) y^\alpha \,\mathrm{d}y = \delta_{0\alpha} \tag{104}$$

The conditions (104) can be translated into the condition in the dual Fourier space; they are

$$D^\alpha_\zeta \hat{\lambda}(0) = \delta_{0\alpha} \tag{105}$$

or

$$\hat{\lambda}(0) = 1 + \mathcal{O}(\zeta^{|\alpha|+1}) \tag{106}$$

In what follows, we shall show how the finite element–moving least-square reproducing kernel formula meets the m-consistency criterion.

PROPOSITION 3.1 (A genuine MLSRK formula). The following FE–MLSRK interpolation,

$$\mathcal{F}^{m,k}_{\varrho,h} u = \sum_{j\in\mathbf{Z}^n} \left(\int_{\mathbf{R}^n} \mathcal{K}_\varrho(y-x_j)u(y)\,\mathrm{d}y\right) \check{\mathcal{K}}^h_\varrho(x-x_j,x) w_j \tag{107}$$

is an L_p-approximation.

PROOF. It suffices to show that

$$\begin{cases} \mathcal{F}^{m,k}_{\varrho,h} y^\alpha = x^\alpha \\ D^\beta_x \mathcal{F}^{m,k}_{\varrho,h} y^\alpha = \dfrac{\alpha!}{(\alpha-\beta)!} x^{\alpha-\beta} \quad |\alpha|, |\beta| \leqslant m \end{cases} \tag{108}$$

(i) By using Eqs. (75) and (77), it is straightforward that

$$\begin{aligned} \mathcal{F}^{m,k}_{\varrho,h} y^\alpha &= \sum_{j\in\mathbf{Z}^n} \left(\int_{\mathbf{R}^n} \mathcal{K}_\varrho(y-x_j) y^\alpha \,\mathrm{d}y\right) \check{\mathcal{K}}^h_\varrho(x-x_j) w_j \\ &= \sum_{j\in\mathbf{Z}^n} \check{\mathcal{K}}^h_\varrho(x-x_j) x_j^\alpha w_j = x^\alpha \end{aligned} \tag{109}$$

(ii)
$$D_x^\beta\left(\mathcal{F}_{\varrho,h}^{m,k}y^\alpha\right)=\sum_{j\in\mathbf{Z}^n}D_x^\beta\check{\mathcal{K}}_\varrho^h(x-x_j)\left(\int_{\mathbf{R}^n}\mathcal{K}_\varrho(y-x_j)y^\alpha\,\mathrm{d}y\right)w_j$$
$$=\sum_{j\in\mathbf{Z}^n}D_x^\beta\check{\mathcal{K}}_\varrho^h(x-x_j)x_j^\alpha w_j$$
$$=\frac{\alpha!}{(\alpha-\beta)!}x^{\alpha-\beta}\quad\square \tag{110}$$

3.2. Controlled L_p-approximation

The term L_p-approximation is usually used as a synonym of L_p convergence, whereas the term of *controlled L_p-approximation* is a specific concept, which was introduced by Strang [27] in designing robust finite element shape functions in the early 1970s. Recently, there has been a revival interest in the field of approximation theory [12,9,16,17]. One ambition of this work is to embrace the MLSRK method with this notion.

We proceed by defining the scaling operator and discrete convolution,

$$\sigma_h f:=f(x/h) \tag{111}$$
$$f*'g:=\sum_{j\in\mathbf{Z}^n}f(x-j)g(j) \tag{112}$$

The controlled L_p-approximation is defined as follows:

DEFINITION 3.5 (Controlled L_p-approximation). If $u\in W_p^k(\mathbb{R}^n)$, $\phi\in\mathcal{S}_r$, there exist weights c^h such that

$$(i)\quad \|u-\sigma_h(\phi*'c^h)h^{-n/p}\|_{L_p(\Omega)}\leqslant \text{const.}h^k|u|_{W_p^k(\Omega)} \tag{113}$$
$$(ii)\quad \|c^h\|_{\ell^p(\mathbf{Z}^n)}\leqslant \text{const.}\|u\|_{L_p(\Omega)} \tag{114}$$

It should be noted that for exact sampling interpolation $c_j^h=u_j^h$.

In 1973, Strang and Fix proved the following theorem in their seminal paper [26]. The original proof is made for a general case of multiple scaling window functions. Here, we only state the case of a single scaling function.

THEOREM 3.1 ([26]). Let $\phi\in W_{2,c}^{k-1}(\mathbb{R}^n)$. Then the following are equivalent:

(i) $\forall\ |\alpha|<k$,

$$(ia)\quad \hat\phi(0)=1,\ \hat\phi(2\pi j)=0,\quad \forall j\in\mathbf{Z}^n\backslash\{0\} \tag{115}$$
$$(ib)\quad D_\xi^\alpha\hat\phi(2\pi j)=0,\quad \forall j\in\mathbf{Z}^n \tag{116}$$

(ii)

$$\sum_{j\in\mathbf{Z}^n}\phi(x/h-j)j^\alpha=x^\alpha \tag{117}$$

(iii) For each $u\in W_2^k(\mathbb{R}^n)$ there exists weights such that

$$(iiia)\quad \|u-\sigma_h(\phi*'c^h)h^{-n/p}\|_{L_p(\Omega)}\leqslant \text{const.}h^k|u|_{W_p^k(\Omega)} \tag{118}$$
$$(iiib)\quad \|c^h\|_{\ell^p(\mathbf{Z}^n)}\leqslant \text{const.}\|u\|_{L_p(\Omega)} \tag{119}$$

Note that the condition (i) was later relaxed into the standard Strang–Fix condition (43) (see de Boor and Jia [9] for discussion).

It should be noted that the characterization of the L_p-approximation in Strang and Fix's original paper is not quite right, a counterexample was given by Jia [15]. The modified characterization of controlled L_p-approximation is due to de Boor and Jia [9] and Dahmen and Micchelli [11,12]. The detailed procedure is out of the scope of this paper. Nevertheless, the discovery made by Strang and Fix reveals a profound regulation: if a shape function satisfies kth order Strang–Fix condition, it can then generate a kth order controlled L_p-approximation, vice versa. And, this will be our departure point.

As will be shown later, since the MLSRK approximation is a 'controlled' L_p-approximation, the kernel function does satisfy specific Strang–Fix conditions. However, the real purpose of this work is not to claim that the MLSRK algorithm is a controlled L_p-approximation, but to use the notion of controlled L_p-approximation to enhance the computational capacity of the MLSRK algorithm. To be precise, instead of asking what kind Strang–Fix condition that the kernel function should meet, our question is: what is the order of the Strang–Fix condition that the window function should meet? and how it affects the behaviors of the kernel function which it generates. This is vitally important even for the problem with non-uniform particle distribution, since the order of the convergence rate of a uniform background particle distribution provides adequate information for the interior error estimate. To us, the meaning of controlled L_p-approximation is really two-fold: first, as its original interpretation, the discrete norm of the weighted coefficients is controlled by the L_p norm of the approximating function; second, as Theorem 3.1 states, the power of the convergence rate is characterized, or controlled by the order of the Strang–Fix condition that the scaling function satisfies. This does, in particular, make a lot of sense for constructing a powerful MLSRK function. This is another type of control problem; i.e. by manipulating and controlling the Fourier transform of the window function (not the kernel function!), one is able to control the convergence rate of the corresponding MLSRK approximation—a controlled L_p-approximation. We shall devote Section 5 to illustrate our theory. Before we get more involved, it may be expedient to find out first what characterization the MLSRK method possesses as a L_p approximation.

The following statement is an adaptation of the similar results of Schoenberg [24,25] and Aubin [2] under the different scenario.

THEOREM 3.2. The m-order moving least-square reproducing kernel approximation satisfies

$$(ia) \quad \|u - \mathcal{R}^m_{\varrho,h} u\|_{H^k} \leqslant C_1 \varrho^{m+1-k} \|u\|_{H^{m+1}}, \quad \forall k = 0,1,\dots,m \tag{120}$$

$$(ib) \quad \|u^h\|_{\ell^2(\mathbf{Z}^n)} \leqslant C_2 \|u\|_{L^2(\mathbf{R}^n)} \tag{121}$$

where $\|u^h\|_{\ell(\mathbf{Z}^n)} := h^{n/2}(\sum_{j\in\mathbf{Z}^n}(u^h_j)^2)^{1/2}$, *if one of the following sets conditions are satisfied,*

$$\begin{cases} (iia) \quad \displaystyle\sum_{j\in\mathbf{Z}^n}(x_j - x)^\alpha \mathcal{K}^h_\varrho(x_j - x, x) w_j = \delta_{0\alpha} \\ (iib) \quad \displaystyle\sum_{j\in\mathbf{Z}^n}(x_j - x)^\alpha D^\beta_x \mathcal{K}^h_\varrho(x_j - x, x) w_j = \alpha! \delta_{\alpha\beta} \end{cases} \tag{122}$$

$$\begin{cases} (iiia) \quad \hat{\mathcal{K}}^h_\varrho(0,x) = 1, \qquad \hat{\mathcal{K}}_\varrho(2\pi j a, x) = 0\ , \quad j \in \mathbf{Z}^n\backslash\{0\} \\ (iiib) \quad D^\alpha_\xi \hat{\mathcal{K}}^h(2\pi j a, x) = 0, \quad j \in \mathbf{Z}^n \quad a \in \mathbb{R}^n \quad 1 \leqslant |\alpha|, |\beta| \leqslant m \end{cases} \tag{123}$$

where $a := \varrho/h > 0$ *is the dilation coefficient.*

PROOF. In Part I, we have shown (ii) $\Rightarrow$ (i). Therefore, here we only need to show (ii) $\Longleftrightarrow$ (iii).

We first show (ii) $\Rightarrow$ (iii). Applying the Poisson summation formula (37) to (iia) and letting $w_j = h^n$, one has

$$\sum_{j\in Z^n}(x_j-x)^\alpha \mathcal{K}_\varrho^h(x_j-x,x)h^n=\sum_{j\in Z^n}\exp\left[-\mathrm{i}\frac{2\pi j}{h}x\right](i)^\alpha D_\zeta^\alpha\hat{\mathcal{K}}_\varrho^h\left(\frac{2\pi j}{h},x\right)$$
$$=(i)^\alpha D_\zeta^\alpha\hat{\mathcal{K}}_\varrho^h(0,x)+\sum_{j\in Z^n\backslash\{0\}}(i)^\alpha\exp\left[-\mathrm{i}\frac{2\pi j}{h}x\right]D_\zeta^\alpha\hat{\mathcal{K}}_\varrho^h\left(\frac{2\pi j}{h},x\right)$$
$$=\delta_{0\alpha} \tag{124}$$

Since $\{\exp(-\mathrm{i}(2\pi j)/h\,x)\}$ are linear independent, it yields

$$D_\zeta^\alpha\hat{\mathcal{K}}_\varrho^h\left(\frac{2\pi j}{h},x\right)=0,\quad \forall j\in Z^n\backslash\{0\} \tag{125}$$

$$D_\zeta^\alpha\hat{\mathcal{K}}_\varrho^h\left(2\pi j\frac{\varrho}{h},x\right)=D_\zeta^\alpha\hat{\mathcal{K}}^h(2\pi ja,x)=0,\quad \forall\, j\in Z^n\backslash\{0\} \tag{126}$$

and

$$D_\zeta^\alpha\hat{\mathcal{K}}_\varrho^h(0,x)=D_\zeta^\alpha\hat{\mathcal{K}}^h(0,x)=\delta_{0\alpha} \tag{127}$$

We now show (iii) $\Rightarrow$ (ii).

(1) (iii) $\rightarrow$ (iia)

By the Poisson summation formula (37), it is straightforward that

$$\sum_{j\in Z^n}(x_j-x)^\alpha\hat{\mathcal{K}}_\varrho^h\left(\frac{x_j-x}{\varrho},x\right)h^n=D_\zeta^\alpha\hat{\mathcal{K}}_\varrho^h(0,x)=\delta_{0\alpha} \tag{128}$$

(2) (iii) $\rightarrow$ (iib)

Applying the Poisson summation formula (37) to (iib) yields

$$\sum_{j\in Z^n}(x_j-x)^\alpha D_x^\beta\hat{\mathcal{K}}_\varrho^h(x_j-x,x)h^n$$
$$=\sum_{j\in Z^n}(i)^\alpha D_\zeta^\alpha((-\mathrm{i}\zeta)^\beta\hat{\mathcal{K}}_\varrho^h(\zeta,x))|_{\zeta=\frac{2\pi j}{h}}\exp\left(-\mathrm{i}\frac{2\pi j}{h}x\right)$$
$$=(i)^{|\alpha|-|\beta|}\sum_{j\in Z^n}\left\{\sum_{|\gamma|\leqslant|\alpha|}\binom{\alpha}{\gamma}\left(D_\zeta^{\alpha-\gamma}\zeta^\beta\right)\left(D_\zeta^\gamma\hat{\mathcal{K}}_\varrho^h(\zeta,x)\right)\right\}\Bigg|_{\zeta=\frac{2\pi j}{h}}\exp\left(-\mathrm{i}\frac{2\pi j}{h}x\right)$$
$$=(i)^{|\alpha|-|\beta|}\sum_{|\gamma|\leqslant|\alpha|}\binom{\alpha}{\gamma}\left(D_\zeta^{\alpha-\gamma}\zeta^\beta\right)\left(D_\zeta^\gamma\hat{\mathcal{K}}_\varrho^h(\zeta,x)\right)\Bigg|_{\zeta=0}$$
$$+(i)^{|\alpha|-|\beta|}\sum_{j\in Z^n\backslash\{0\}}\left\{\sum_{|\gamma|\leqslant|\alpha|}\binom{\alpha}{\gamma}\left(D_\zeta^{\alpha-\gamma}\zeta^\beta\right)\left(D_\zeta^\gamma\hat{\mathcal{K}}_\varrho^h(\zeta,x)\right)\right\}\Bigg|_{\zeta=\frac{2\pi j}{h}}\exp\left(-\mathrm{i}\frac{2\pi j}{h}x\right)$$
$$=(i)^{|\alpha|-|\beta|}\frac{\alpha!}{(\alpha-\gamma)!\gamma!}\beta!D_\zeta^\gamma\hat{\mathcal{K}}_\varrho^h(\zeta,x)\Bigg|_{\zeta=0}\Leftarrow \text{ by (iiib)} \tag{129}$$

where $\gamma=\alpha-\beta$. One may find that Eq. (129) is not equal to zero, if only $\gamma=0$. Hence,

$$(i)^{|\alpha|-|\beta|}\frac{\alpha!\beta!}{(\alpha-\beta)!\beta!}\delta_{0(\alpha-\beta)}=\alpha!\delta_{\alpha\beta} \tag{130}$$

It follows that

$$\sum_{j\in Z^n}(x_j-x)^\alpha D_x^\beta\mathcal{K}_\varrho^h(x_j-x,x)h^n=\alpha!\delta_{\alpha\beta}\quad\square \tag{131}$$

REMARK 3.2. (i) The above theorem shows that the m-order moving least-square reproducing kernel function satisfies

$$\mathcal{K}^h(\cdot,x)\in\mathcal{SF}_{1/a}^{(m)} \tag{132}$$

However, there is no explicit requirement for generating window function. Meyer [22] has shown that if $\phi \in \mathcal{S}_r$ then at least $\phi \in \mathcal{SF}^{(r)}$. The tricky point here is that even if $\phi \in \mathcal{SF}^{(r)}$ and $r > m$, it may not be true that $\phi \in \mathcal{SF}^{(m)}_{1/a}$; that brings the second point.

(ii) In the moving least-square reproducing kernel formulation, the invariant translation unit ϱ is not necessarily equal to the unit of sampling interval, h. As a matter of fact,

$$\varrho := ah\ , \quad \text{and} \quad a \geqslant 1 \tag{133}$$

4. Multiresolution approximation

4.1. Riesz basis and numerical stability

As claimed in [19] and Part I of this work, the MLSRK approximation is a multiresolution analysis in $C^m(\mathbb{R}^n)$, though argument presented there was intuitive. Rigorously speaking, if a moving least-square reproducing kernel function generates a multiresolution analysis, it must satisfy one of the following conditions.

THEOREM 4.1 ([5]). For any function $\phi \in L^2(\mathbb{R}^n)$ and constants $0 < A \leqslant B < \infty$, the following two statements are equivalent.

(i) $\{\phi(\cdot - k) : k \in \mathbf{Z}^n\}$ satisfies the Riesz condition with Riesz bounds A and B; that is for any $\{c_k\} \in \ell^2(\mathbf{Z}^n)$

$$A\|\{c_k\}\|^2_{\ell^2(\mathbf{Z}^n)} \leqslant \left\| \sum_{k\in\mathbf{Z}^n} c_k \phi(\cdots, -k) \right\|^2_{L^2(\mathbf{R}^n)} \leqslant B\|\{c_k\}\|^2_{\ell^2(\mathbf{Z}^n)} \tag{134}$$

(ii) The Fourier transform $\hat{\phi}$ of ϕ satisfies

$$A \leqslant \sum_{j\in\mathbf{Z}^n} |\hat{\phi}(\theta + 2\pi j)|^2 \leqslant B, \quad a.e \tag{135}$$

where $\theta \in \mathbb{R}^n$.

One might think that the above condition is purely a mathematical restriction without too much practical implication. On the contrary, the above condition is the major concern of the numerical stability of the practical algorithms.

For example, if function ϕ generates a set of finite element basis functions, the fundamental Gram matrix generated by this particular set of finite element shape functions will be

$$G_h := \{g^h_{ij}\} \tag{136}$$

where

$$g^h_{ij} := \int_{\mathbf{R}^n} \phi_h(x_i - x)\phi_h(x_j - x)\,\mathrm{d}x \tag{137}$$

Fix and Strang [13] have shown that the condition number of the Gram matrix (136) can be expressed as

$$\kappa(G_h) := \frac{\sup\limits_{\theta\in\mathbf{R}^n} \sum\limits_{j\in\mathbf{Z}^n} |\hat{\phi}(\theta + 2\pi j)|^2}{\inf\limits_{\theta\in\mathbf{R}^n} \sum\limits_{j\in\mathbf{Z}^n} |\hat{\phi}(\theta + 2\pi j)|^2} \tag{138}$$

Obviously, the denominator, $\inf_{\theta\in\mathbf{R}^n} \sum_{j\in\mathbf{Z}^n} |\hat{\phi}(\theta + 2\pi j)|^2$, cannot be zero, otherwise, the condition number $\kappa(G_h)$ will be unbounded. The condition is further shown by Fix and Strang [13] to be equivalent to the other two conditions:

THEOREM 4.2 ([13]). Suppose $\theta_0 \in \mathbb{R}^n$, then the following stability conditions on ϕ are equivalent

$$(i)\quad \hat{\phi}(2\pi j + \theta_0) = 0 \quad \forall j \in \mathbf{Z}^n \tag{139}$$

$$(ii)\quad \sum_{j\in\mathbf{Z}^n} \exp(-\mathrm{i}j\theta_0)\phi_h^j \equiv 0 \tag{140}$$

$$(iii)\quad \sum_{j\in\mathbf{Z}^n} |\hat{\phi}(2\pi j + \theta_0)|^2 = 0 \tag{141}$$

The point here is that the stability condition imposed on the basis functions is the same as the Riesz bound condition, which is required by a multiresolution analysis. In other words, if the MLSRK is a multiresolution analysis, it automatically satisfy the stability condition; vice versa, if the kernel function family satisfies the stability condition (4.2), it generates a multiresolution analysis.

For our MLSRK formulation, the Fix–Strang stability condition should be slightly modified as: there exists no $\theta_0 \in \mathbb{R}^n$ such that one of the following conditions holds

$$(i)\quad \hat{\mathcal{K}}(2\pi j a + a\theta_0) = 0\,, \quad \forall j \in \mathbf{Z}^n, \tag{142}$$

$$(ii)\quad \sum_{j\in\mathbf{Z}^n} \exp(-\mathrm{i}x_j\theta_0)\,\frac{1}{\varrho^n}\,\mathcal{K}\left(\frac{x_j - x}{\varrho}\right) h^n = 0 \tag{143}$$

$$(iii)\quad \sum_{j\in\mathbf{Z}^n} |\hat{\mathcal{K}}(2\pi j a + a\theta_0)|^2 = 0 \tag{144}$$

where $a = h/\varrho$.

In what follows, we shall show that for the MLSRK function, the above stability condition is satisfied, provided that the particle distribution is reasonably dense.

THEOREM 4.3. Assume that ϕ is compact supported, and the FE–MLSRK approximation $\mathcal{F}^{m,0}_{\varrho,h} := \mathcal{R}^m_{\varrho,h} : C(\mathbb{R}^n) \mapsto C^m(\mathbb{R}^n)$ is non-degenerated, then, the corresponding reproducing kernel function always satisfies the Riesz bound condition—stability condition.

$$A \leqslant \sum_{j\in\mathbf{Z}^n} \left| \hat{\mathcal{K}}(2\pi j a + a\theta_0) \right|^2 \leqslant B, \quad \theta_0 \in \mathbb{R}^n \tag{145}$$

PROOF. We first show the left inequality. By the Taylor expansion,

$$\begin{aligned}\exp(\mathrm{i}x_j\theta) = {} & \exp(\mathrm{i}x\theta) + \frac{(rm\mathrm{i}\theta)^{\alpha_1}}{\alpha_1!}\exp(\mathrm{i}x\theta)(x_j - x)^{\alpha_1} + \cdots + \frac{(rm\mathrm{i}\theta)^{\alpha_{\ell-1}}}{\alpha_{\ell-1}!}\exp(\mathrm{i}x\theta)(x_j - x)^{\alpha_{\ell-1}} \\ & + \sum_{|\alpha_\ell|=m+1} \frac{(\mathrm{i}\theta)^{\alpha_\ell}}{\alpha_\ell!}\exp(\mathrm{i}x_\xi\theta)(x_j - x)^{\alpha_\ell}\end{aligned} \tag{146}$$

where $x_\xi = x + \eta(x_j - x)$ and $|\eta| < 1$. By (76), it follows that

$$\begin{aligned}& \sum_{j\in\mathbf{Z}^n} \exp(\mathrm{i}x_j\theta)\,\frac{1}{\varrho^n}\,\mathcal{K}\left(\frac{x_j - x}{\varrho}\right) h^n \\ & = \exp(\mathrm{i}x\theta) + \sum_{j\in\mathbf{Z}^n}\sum_{|\alpha_\ell|=m+1} \frac{(\mathrm{i}\theta)^{\alpha_\ell}}{\alpha_\ell!}\exp(\mathrm{i}x_\xi\theta)\,(x_j - x)^{\alpha_\ell}\,\frac{h^n}{\varrho^n}\,\mathcal{K}\left(\frac{x_j - x}{\varrho}\right) \\ & = \exp(\mathrm{i}x\theta)\left\{1 + \varrho^{m+1}\sum_{j\in\mathbf{Z}^n}\sum_{|\alpha_\ell|=m+1} \frac{(\mathrm{i}\theta)^{\alpha_\ell}}{\alpha_\ell!}\exp(\mathrm{i}\theta\,\eta(x_j - x))\left(\frac{x_j - x}{\varrho}\right)^{\alpha_\ell}\frac{h^n}{\varrho^n}\mathcal{K}\left(\frac{x_j - x}{\varrho}\right)\right\}\end{aligned} \tag{147}$$

Since the kernel function $\mathcal{K}(\cdot)$ is compact supported, $\forall x \in \mathbb{R}^n$ there are only finite number of j, such that

$$|x_j - x| \leqslant k\varrho \tag{148}$$

where $k\varrho$ is the characteristic length of the compact support.

Hence,

$$\begin{aligned}&\left|\sum_{j\in \mathbf{Z}^n}\sum_{|\alpha_\ell|=m+1}\frac{(\mathrm{i}\theta)^{\alpha_\ell}}{\alpha_\ell!}\exp\left(\mathrm{i}\theta\,\eta(x_j - x)\right)\left(\frac{x_j - x}{\varrho}\right)^{\alpha_\ell}\frac{h^n}{\varrho^n}\mathcal{K}\left(\frac{x_j - x}{\varrho}\right)\right| \\ &\leqslant C_\alpha \sum_{j\in\mathbf{Z}^n}\left|\left(\frac{x_j - x}{\varrho}\right)^{\alpha_\ell}\mathcal{K}\left(\frac{x_j - x}{\varrho}\right)\frac{h^n}{\varrho^n}\right| \\ &\leqslant C_\ell\end{aligned} \tag{149}$$

Consequently, if $|\varrho|^{m+1}C_\ell < 1$, one will have

$$\sum_{j\in\mathbf{Z}^n}\exp\left(\mathrm{i}x_j\theta\right)\frac{h^n}{\varrho^n}\mathcal{K}\left(\frac{x_j - x}{\varrho}\right) \sim \exp(\mathrm{i}x\theta) \neq 0\,. \tag{150}$$

Let $\theta = -\theta_0$. By using the Poisson summation formula (37) and the frequency shifting property of Fourier transform (5) and the scaling property of Fourier transform (8), one has

$$\begin{aligned}\sum_{j\in\mathbf{Z}^n}\exp(-\mathrm{i}x_j\theta_0)\frac{h^n}{\varrho^n}\mathcal{K}\left(\frac{x_j - x}{\varrho}\right) &= \sum_{j\in\mathbf{Z}^n}\hat{\mathcal{K}}(2\pi ja + a\theta_0)\exp(-\mathrm{i}2\pi jax) \\ &\neq 0 \leftarrow \text{ by Eq. (150)}\end{aligned} \tag{151}$$

It follows that $\exists\, J \in \mathbf{Z}^n$, such that

$$\hat{\mathcal{K}}\,(2\pi Ja + a\theta_0) \neq 0, \quad \forall\, \theta_0 \in \mathbb{R}^n \tag{152}$$

then let

$$A = \inf_{\theta_0\in\mathbf{R}^n}\sum_{j\in\mathbf{Z}^n}|\,\hat{\mathcal{K}}(2\pi ja + a\theta_0)\,|^2 = \inf_{\theta_0\in\mathbf{R}^n}\sum_{J}|\hat{\mathcal{K}}(2\pi Ja + a\theta_0)\,|^2 > 0 \tag{153}$$

On the other hand, since the window function $\mathcal{K}$ is compact supported, by the Bessel's inequality

$$\begin{aligned}\sum_{j\in\mathbf{Z}^n}|\,\hat{\mathcal{K}}\,(2\pi ja + a\theta_0)\,|^2 &\leqslant \frac{C}{(2k\varrho)^n}\int_{[-k\varrho,k\varrho]^n}\mathcal{K}(x)^2\,\mathrm{d}x \\ &\leqslant C_1\int_{\mathbf{R}^n}\mathcal{K}(x)^2\,\mathrm{d}x = C_1\|\mathcal{K}\|^2_{L_2(\mathbf{R}^n)} \\ &\leqslant C_b\end{aligned} \tag{154}$$

Let $B = C_b$. We have just shown

$$\sum_{j\in\mathbf{Z}^n}|\,\hat{\mathcal{K}}(2\pi ja + a\theta_0)\,|^2 \leqslant B \quad \square \tag{155}$$

5. Synchronized convergence

5.1. *How to make $C^h_\varrho = 1$?*

The convergence results of MLSRK obtained in Part I of this work [21] is not quite satisfactory in the sense that there are conspicuous discrepancies between the theoretical prediction and the numerical experiments. It is true that this is a complicated matter, especially, in our formulation, there is a distinction between the dilation interval and the sampling interval.

In fact, the actual convergence results obtained in numerical experiments show a higher convergence rate than the theoretical prediction. In Part I, we implicitly admit that the order of approximation of MLSRK is the same as that of the finite element method, if its shape function contains the same order piecewise polynomials. This implies that the window function has nothing to do with the convergence rate, and it only serves as localization instrument. This is a typical stereotyped thinking, which is influenced by the conventional wisdom of traditional finite element method; i.e. we take the following proposition for granted: if one wishes to increase the approximation order, one has to increase the order of the polynomial basis—the central idea behind the p-finite element. Is this still a universally true statement, even in the MLSRK formulation?—our challenge starts from here.

In the MLSRK formulation, the kernel function has the form

$$\mathcal{K}_\varrho(y-x) = \mathcal{C}_\varrho(y-x)\phi_\varrho(y-x) \tag{156}$$

or

$$\mathcal{K}_\varrho^h(x_j-x,x) = \mathcal{C}_\varrho^h(x_j-x,x)\phi_\varrho(x_j-x) \tag{157}$$

where $\mathcal{C}_\varrho^h(y-x,x)$ is the so-called correction function (see Appendix A for details).

One basic requirement on the window function is that $\phi \in C_c^s(\Omega)$ $s >> m$ or $\phi \in \mathcal{S}_r \cap C_c^m(\Omega)$, $r >> m$ at least, where m is approximation order of the MLSRK algorithm.

Apparently, if

$$\mathcal{C}_\varrho^h(y-x,x) = 1 \tag{158}$$

in the interior of Ω. Then, suddenly,

$$\mathcal{K}_\varrho^h(y-x,x) = \phi_\varrho(y-x) \tag{159}$$

By doing this, we liberate the window function from the nest of the polynomial basis, which essentially lift the kernel function to a higher-order approximation level. The following theorem tells us one way to do it.

THEOREM 5.1 (A sufficient condition for $\mathcal{C}_\varrho^h(y-x,x)=1$). For the m-order MLSRK function, if window function $\phi(\cdot) \in \mathcal{SF}_{1/a}^{(m)}$, and

$$D_\xi^\alpha \hat{\phi}(0) = 0, \quad \forall 1 \leqslant |\alpha| \leqslant m \tag{160}$$

the corresponding correction function

$$\mathcal{C}_\varrho^h(y-x,x) = 1 \tag{161}$$

with respect to the uniform mesh $h\boldsymbol{Z}^n$.

PROOF. The proof is straightforward. By the assumption (160), if $1 \leqslant |\alpha_i| \leqslant m$,

$$\begin{aligned} m_{\alpha_i}^h(x) &= \sum_{j\in \boldsymbol{Z}^n} \frac{h^n}{\varrho^n}\left(\frac{x_j-x}{\varrho}\right)^{\alpha_i} \phi\left(\frac{x_j-x}{\varrho}\right) \\ &= \sum_{j\in \boldsymbol{Z}^n} D_\xi^{\alpha_i}\hat{\phi}(2\pi j a)\exp\left(-\mathrm{i}\,\frac{2\pi x}{h}\right) \\ &= D_\xi^{\alpha_i}\hat{\phi}(0) = 0 \end{aligned} \tag{162}$$

Thus, the moments matrix will have the form

$$\boldsymbol{M}_\varrho = \begin{pmatrix} m_0 & 0 & \cdots & 0 \\ 0 & m_{2\alpha_1} & \cdots & m_{\alpha_\ell} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & m_{\alpha_\ell} & \cdots & m_{2\alpha_{\ell-1}} \end{pmatrix} \tag{163}$$

It is then easy to verify that the cofactors of moment matrix $\boldsymbol{M}_\varrho$ satisfy

$$A_{1i}(x) = 0, \quad \forall 2 \leqslant \mathrm{i} \leqslant \ell \tag{164}$$

Considering the fact that $m_0 = \hat{\phi}(0) = 1$,

$$\begin{aligned} C^h_\varrho(y-x,x) &= \frac{1}{D_\ell}\left[A_{11}(x) - \left(\frac{y-x}{\varrho}\right)^{\alpha_1} A_{12}(x) + \cdots + (-1)^\ell \left(\frac{y-x}{\varrho}\right)^{\alpha_{\ell-1}} A_{1\ell}(x)\right] \\ &= \frac{A_{11}(x)}{D_\ell} = \frac{1}{m_0} = 1 \quad \square \end{aligned} \tag{165}$$

In what follows, we shall show how such class of window functions can be constructed in a systematic manner.

EXAMPLE 5.1. Consider the kernel function that is generated by a linear polynomial basis and choose cubic B-spline function as the choice of the window function. The Fourier transform of the cubic spline is

$$\hat{\phi}_3(\zeta) = \left(\frac{\sin(\zeta/2)}{\zeta/2}\right)^4 \tag{166}$$

If $a := \varrho/h \in \boldsymbol{Z}^n_+$, it is always true that

$$\phi_3(\cdot) \in \mathcal{SF}^{(m)} \tag{167}$$

because in this case $m = 1 < 3$. Since

$$\begin{aligned} &\hat{\phi}_3(\zeta) = 1 - \frac{1}{6}\zeta^2 + \mathcal{O}(\zeta^3) \\ &\Rightarrow \frac{\mathrm{d}}{\mathrm{d}\zeta}\hat{\phi}_3(0) = 0 \end{aligned} \tag{168}$$

Hence, the cubic spline function satisfies all the conditions in Theorem 5.1, which means that if one chooses the cubic spline as the window function to generate the linear polynomial based kernel function, the corresponding correction function in the interior will be equal to one. One can verify that the higher-order (than cubic) B-spline functions can all be the window function candidate in this particular case.

Before we go on to illustrate our next example, we would like to state a key lemma. Several equivalent forms of the lemma are stated or proved by various authors [11,16,27].

LEMMA 5.1. If $\phi \in \mathcal{SF}^{(m)}$, there exists a constant sequence b supported on $\mathbf{N}^n_m := \{j \mid j \in \boldsymbol{Z}^n;\ |j| \leqslant m,\ \mathrm{card}(\mathbf{N}^n_m) \leqslant n\cdot(m+1)\}$ such that the mapping $\psi := \phi *' b$ satisfies

$$\psi(x) \in \mathcal{SF}^{(m)} \tag{169}$$

and

$$D^\alpha_\zeta \psi(0) = 0, \quad 1 \leqslant |\alpha| \leqslant m \tag{170}$$

or equivalently, ψ reproduces polynominals up to $|\alpha| = m$ order, i.e.

$$\sum_{j\in \boldsymbol{Z}^n} \psi *' (j)^\alpha = x^\alpha \tag{171}$$

The construction of the following proof is basically Strang's idea [27], which may be credited to Babuška [3] as well.

PROOF. Here, we only prove the lemma for one dimensional case; the multi-dimensional extension is straightforward. Let

$$\psi(x) = \sum_{j\in \mathbf{N}_m} \phi(x-j)b_j\,, \qquad \mathbb{N}_m := \{j | j \in \mathbf{Z};\ |j| \leqslant m,\ \mathrm{card}(\mathbb{N}_m) \leqslant m+1\} \tag{172}$$

Then, by the spatial shifting property of the Fourier transform (4), one has

$$\hat{\psi}(\zeta) = \sum_{j\in \mathbf{N}_m} \hat{\phi}(\zeta)\exp(-\mathrm{i}\zeta j) b_j \tag{173}$$

Consequently

$$D^\alpha_\zeta \hat{\phi}(\zeta)|_{\zeta=2\pi j a} = 0 \Longrightarrow D^\alpha_\zeta \hat{\psi}(\zeta)|_{\zeta=2\pi j a} = 0 \tag{174}$$

It remains to show that

$$\hat{\psi}(0) = 1 \tag{175}$$

$$D^n_\zeta \hat{\psi}(0) = 0, \qquad 1 \leqslant n \leqslant m \tag{176}$$

Without losing generality, assume

$$\hat{\phi}(\zeta) = 1 + \alpha_1\zeta + \alpha_2\zeta^2 + \cdots + \alpha_m\zeta^m + \mathcal{O}(\zeta^{m+1}) \tag{177}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_m$ are not all zero (otherwise we have done!).

We can choose $\{b_j\}$ in such a way to enforce the condition

$$D^\alpha_\zeta \hat{\psi}(0) = \delta_{0\alpha} \tag{178}$$

Substituting (173) into Eq. (178) yields the following algebraic equations,

$$\sum_{j\in \mathbf{N}_m} b_j = 1 \tag{179}$$

$$\sum_{j\in \mathbf{N}_m} (-ij)b_j = -\alpha_1 \tag{180}$$

$$\sum_{j\in \mathbf{N}_m} \left(2\alpha_1(-ij) + (-ij)^2\right) b_j = -2!\alpha_2 \tag{181}$$

$$\cdots\cdots\cdots$$

$$\sum_{j\in \mathbf{N}_m} \left(\frac{m!}{1!}\,\alpha_{m-1}(-ij) + \frac{m!}{2!}\,\alpha_{m-2}(-ij)^2 + \cdots + \frac{m!}{m!}\,(-ij)^m\right) b_j = -m!\alpha_m \tag{182}$$

Since $\{1, x, x^2, \ldots, x^m\}$ are independent, the coefficient matrix of the above algebraic equation is always nonsingular. We then conclude that the sequence $\{b_j\}$ is uniquely determined, and it automatically yields Eqs. (175) and (176). For simplicity, let $a = 1$, then

$$\begin{aligned}\sum_{j\in \mathbf{N}_m} (x-j)^\alpha \psi(x-j) &= \sum_{j\in \mathbf{N}_m} D^\alpha_\zeta \hat{\psi}(2\pi j)\exp\{-\mathrm{i}2\pi j x\} \\ &= D^\alpha_\zeta \hat{\psi}(0) = \delta_{0\alpha}\end{aligned} \tag{183}$$

Consequently,

$$\begin{aligned}\sum_{j\in \mathbf{Z}} j^\alpha \psi(j-x) &= \sum_{j\in \mathbf{Z}} ((j-x)+x)^\alpha\, \psi(j-x) \\ &= \sum_{j\in \mathbf{Z}} \sum_{\gamma\leqslant\alpha} \binom{\alpha}{\gamma} (j-x)^{\alpha-\gamma} x^\gamma \psi(j-x) \\ &= x^\gamma \binom{\alpha}{\gamma} \delta_{0(\alpha-\gamma)} \\ &= x^\alpha \quad \square\end{aligned} \tag{184}$$

EXAMPLE 5.2. For the kernel function with a cubic polynomial basis, in order to enforce the correction function to be equal to 1, the window function candidate has to satisfy the following conditions:

$$\phi \in \mathcal{SF}^{(3)} \quad \text{and} \quad \frac{\mathrm{d}^n}{\mathrm{d}\zeta^n}\hat{\phi}(0) = 0 \quad 1 \leqslant n \leqslant 3 \tag{185}$$

Obviously, the B-spline function family is no longer satisfying the requirements, since $\mathrm{d}^2/\mathrm{d}\zeta^2\hat{\phi}(0) \neq 0$. Nevertheless, we can modify the B-spline function based on Lemma (5.1) such that the modified window function can meet the requirements. For B-spline functions, there is a smart, easy procedure to follow, which was devised by Schenoberg in his celebrated papers [24,25], so that we do not need to solve the algebraic equations (179)–(182). Let the fifth-order B-spline function be the pre-choice window function (there is no advantage of doing this by using cubic spline). The Fourier transform of the fifth order B-spline function is

$$\hat{\phi}_5(\zeta) = \left(\frac{\sin\zeta/2}{\zeta/2}\right)^6 \tag{186}$$

and

$$\hat{\phi}_5(\zeta) = 1 - \frac{1}{4}\zeta^2 + \mathcal{O}(\zeta^4) \tag{187}$$

One can see that

$$\frac{\mathrm{d}^2}{\mathrm{d}\zeta^2}\hat{\phi}_5(0) \neq 0 \tag{188}$$

To remedy the situation, set

$$\psi(x) = \sum_j \phi_5(x-j)b_j \tag{189}$$

The Fourier transform of $\psi(x)$ takes the form

$$\hat{\psi}(\zeta) = \sum_j \hat{\phi}_5(\zeta)\exp(-\mathrm{i}\zeta j)b_j \tag{190}$$

since

$$\frac{3}{2} - \frac{\cos\zeta}{2} = 1 + \frac{1}{4}\zeta^2 + \mathcal{O}(\zeta^4) \tag{191}$$

we can choose

$$\hat{\psi}(\zeta) = \hat{\phi}_5(\zeta)\left(\frac{3}{2} - \frac{\cos\zeta}{2}\right) \tag{192}$$

$$\Rightarrow \hat{\psi}(\zeta) = 1 + \mathcal{O}(\zeta^4) \tag{193}$$

Thus, $\psi(x)$ meets all the requirements for $C_\varrho^h(y-x,x) = 1$ in the interior domain. On the other hand,

$$\hat{\psi}(\zeta) = \sum_j \hat{\phi}_5(\zeta)\exp(-\mathrm{i}\zeta j)b_j = \hat{\phi}_5(\zeta)\left(\frac{3}{2} - \frac{\cos\zeta}{2}\right) \tag{194}$$

which leads to

$$b_{-1} = -\frac{1}{4}, \qquad b_0 = \frac{3}{2}, \qquad b_1 = -\frac{1}{4} \tag{195}$$

$$\psi(x) = b_{-1}\phi_5(x-1) + b_0\phi_5(x) + b_1\phi_5(x+1) \tag{196}$$

where

$$\phi_5(x) = \begin{cases} \frac{1}{60}(33 - 30x^2 + 15x^4 - 5x^5), & 0 \leqslant x \leqslant 1 \\ \frac{1}{120}(51 + 75x - 210x^2 + 150x^3 - 45x^4 + 5x^5), & 1 \leqslant x \leqslant 2 \\ \frac{1}{120}(3-x)^5 2, & \leqslant x \leqslant 3 \\ 0, & 3 < x \end{cases} \tag{197}$$

and $\phi_5(-x) = \phi_5(x)$.

The modified new window function is plotted in comparison with the original window function—the fifth-order spline in Fig. 1. In this case, the radius of the compact support increases from 3ϱ to 4ϱ.

EXAMPLE 5.3. In this example, we consider the kernel function that is generated by a quadratic polynomial basis. Readers may notice that our presentation is in a reverse order, i.e. instead of discussing this example first, we discussed the kernel function with cubic polynomial basis first. The reason is following.

In the last example, we have constructed new window function ψ, $\psi \in \mathcal{SF}^{(5)}$ and $\mathrm{d}^n/\mathrm{d}\zeta^n\hat{\psi}(0) = 0$, $n = 1,2,3$; If we use this ψ as our window function to construct the quadratic polynomial based kernel function, the moment matrix will be singular, i.e.

$$\boldsymbol{M} = \begin{pmatrix} m_0 & m_1 = 0 & m_2 = 0 \\ m_1 = 0 & m_2 = 0 & m_3 = 0 \\ m_2 = 0 & m_3 = 0 & m_4 \end{pmatrix} \tag{198}$$

Principally, we can still use the method stated in Lemma 5.1 to construct the qualified window function without causing singularity in moment matrix. In doing this, we have to enforce another condition

$$\mathrm{d}^3/\mathrm{d}\zeta^3\hat{\psi}(0) \neq 0 \tag{199}$$

Taking this into the consideration, we choose

$$\hat{\psi}(\zeta) = \hat{\phi}_5(\zeta)\left[\left(\frac{3}{2} - \frac{\cos\zeta}{2}\right) + \mathrm{i}\,(2\sin\zeta - \sin(2\zeta))\right] \tag{200}$$

$$\Rightarrow \hat{\psi}(\zeta) = 1 + |\zeta|^3 + \mathcal{O}(\zeta^5) \tag{201}$$

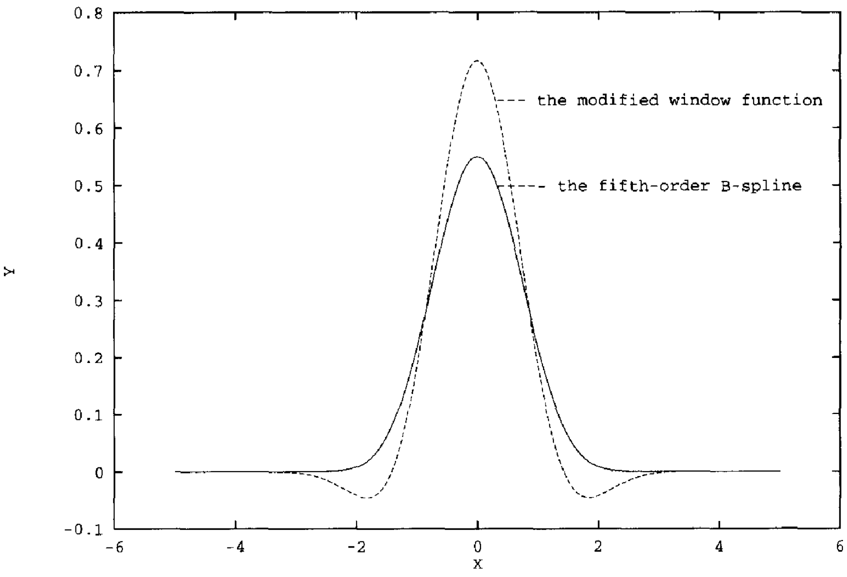

Fig. 1. The original B-spline function and the modified new window function.

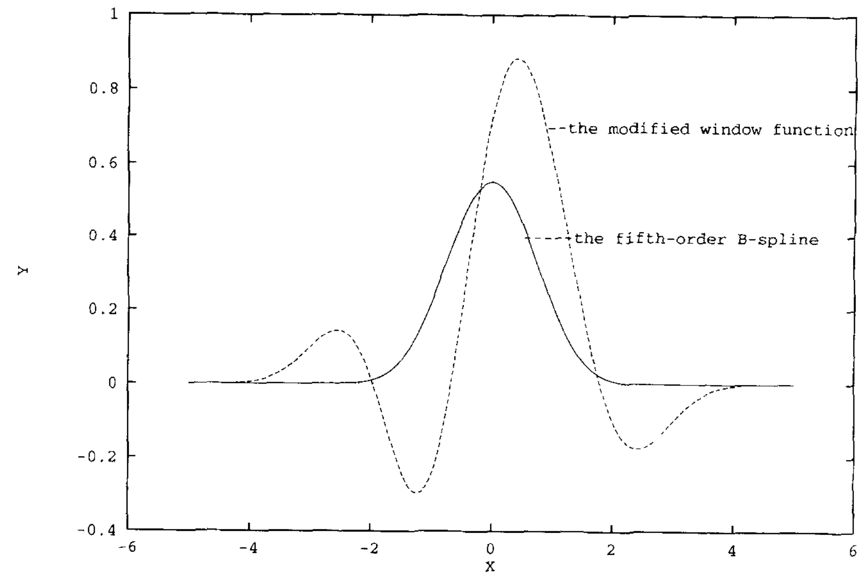

Fig. 2. The original B-spline function and the modified new window function.

After inversion, one may find that

$$b_{-2} = \frac{1}{2}, \qquad b_{-1} = -\frac{5}{4}, \qquad b_0 = \frac{3}{2}, \qquad b_1 = \frac{3}{4}, \qquad b_2 = -\frac{1}{2} \tag{202}$$

$$\psi(x) = \frac{1}{2}\,\phi_5(x-2) - \frac{5}{4}\,\phi_5(x-1) + \frac{3}{2}\,\phi_5(x) + \frac{3}{4}\,\phi_5(x+1) - \frac{1}{2}\,\phi_5(x+2) \tag{203}$$

It should be noted that there is a drawback of the modified window function, i.e. it is no longer symmetric with respect to y-axis.

Again, the newly modified window function $\psi(x)$ is plotted against the original window function—the fifth-order spline. From Fig. 2, one can clearly see that the profile of the modified window function is asymmetric with the y-axis, and the length of the compact support of the modified window function has increased significantly.

5.2. Synchronized convergence

If the correction function is successfully enforced to be 1 in the interior region of a domain. The MLSR kernel function immediately takes the position of the window function in the interior domain. As shown previously in Section 3, the mth order MLSRK function only satisfies $\mathcal{K}_\varrho^h \in \mathcal{SF}_{1/a}^{(m)}$. On the other hand, in order to construct a mth order MLSRK partition of unity, we may require the window function $\phi(\cdot) \in H_0^{m+s}(\mathbb{R}^n)$ or at least $\phi(\cdot) \in \mathcal{S}_r \cap C_0^m(\mathbb{R}^n)$ and $r \geqslant m+s$ for some positive integer $s > 0$. As shown by Meyer [22], the condition $\phi(\cdot) \in \mathcal{S}_r$, $r \geqslant m+s$ is equivalent to $\phi(\cdot) \in \mathcal{SF}^{(m+s)}$ at least.

This means that when we tune the irregular particle distribution into a uniform distribution, there is a leap in the order of Strang–Fix condition for the kernel function. The implication of this development is that there could be an increase in the approximation order of the numerical calculation in the interior domain of the problem.

The primary gain we get from the process is that

$$\mathcal{K}^h \in \mathcal{SF}_{1/a}^{(m)} \Longrightarrow \mathcal{K}^h = \phi \in \mathcal{SF}_{1/a}^{(m+s)} \tag{204}$$

How can we take advantage of the situation? there are two obvious ways:

(i) Using Lemma 5.1 again to reconstruct the kernel function, (not window function this time ! i.e. one need not to recompute the moments), such that

$$\mathrm{d}^n/\mathrm{d}\zeta^n \hat{\mathcal{K}}^h_\varrho(0) = 0\,, \quad n = 1,2,\dots,m,\dots,m+s \tag{205}$$

This is the best that can happen. However, how to match the interior interpolation with the boundary interpolation need further study.

(ii) Doing nothing. Just enjoying the improvement of the solution.

The followings are the main results in this section.

LEMMA 5.2. Suppose the correction function $C^h_\varrho(y-x,x) = 1$, $\forall x \in \mathbb{R}^n$, *and the window function* $\phi \in \mathcal{SF}^{(m+s)}_{1/a}$. *The mth order MLSRK function satisfies*

$$\sum_{j\in \mathbf{Z}^n} (x_j - x)^\alpha D^\beta_x \mathcal{K}^h_\varrho(x_j - x) h^n = \alpha!\,\delta_{\alpha\beta} \tag{206}$$

where $0 \leqslant |\beta| \leqslant \max\{s,m\}$, $\quad 0 \leqslant |\alpha| \leqslant m+s$.

PROOF. In Part I, we have shown that the mth order MLSRK function satisfies the condition (206) up to the order, $|\alpha|, |\beta| \leqslant m$. Hence, here, we only need to show

$$\sum_{j\in \mathbf{Z}^n} (x_j - x)^\alpha D^\beta_x \mathcal{K}^h_\varrho(x_j - x) h^n = \alpha!\,\delta_{\alpha\beta}\,, \quad \forall\,|\beta| \leqslant \max\{s,m\}, |\alpha| \leqslant m+s \tag{207}$$

By the general moment theorem (2.2),

$$\begin{aligned} -\int_{\mathbf{R}^n} (y-x)^\alpha D^\beta_x \mathcal{K}_\varrho(y-x)\,\mathrm{d}y &= -\varrho^{\alpha-\beta} \int_{\mathbf{R}^n} \left(\frac{y-x}{\varrho}\right)^\alpha D^\beta_{x/\varrho} \mathcal{K}_\varrho(y-x)\,\mathrm{d}y \\ &= \varrho^{\alpha-\beta} m^\beta_\alpha \\ &= \varrho^{\alpha-\beta} (i)^{|\alpha+\beta|} \frac{\alpha!}{(\alpha-\beta)!} D^{\alpha-\beta}_\zeta \hat{\mathcal{K}}(0)\langle \alpha - \beta\rangle \end{aligned} \tag{208}$$

For $|\beta| \leqslant \max\{m,s\}$, and $|\alpha| \leqslant m+s$, it is always true that

$$|\alpha - \beta| \leqslant m$$

By Theorem (3.2) (iiia)–(iiib),

$$D^\gamma_\zeta \hat{\mathcal{K}}(0) = D^\gamma_\zeta \hat{\mathcal{K}}^h(0) = \delta_{0\gamma}, \quad |\gamma| \leqslant m \tag{209}$$

Then by substituting α and β into Eq. (208), one can readily verify that

$$\int_{\mathbf{R}^n} (y-x)^\alpha D^\beta_x \mathcal{K}_\varrho(y-x)\,\mathrm{d}y = \alpha!\,\delta_{\alpha\beta}, \qquad \forall|\beta| \leqslant \max\{m,s\}, \quad |\alpha| \leqslant m+s \tag{210}$$

On the other hand, since $C^h_\varrho = 1$,

$$\mathcal{K}^h_\varrho(x) = \phi_\varrho(x) \in \mathcal{SF}^{(m+s)}_h \tag{211}$$

which leads to

$$\begin{aligned} \sum_{j\in \mathbf{Z}^n} (x_j - x)^\alpha D^\beta_x \mathcal{K}^h_\varrho(x_j - x) h^n &= \int_{\mathbf{R}^n} (y-x)^\alpha D^\beta_x \mathcal{K}_\varrho(y-x)\,\mathrm{d}y \\ &= \alpha!\,\delta_{\alpha\beta} \quad \square \end{aligned} \tag{212}$$

The consequence of Lemma 5.2 is the so-called synchronized convergence phenomenon.

THEOREM 5.2 (Synchronized convergence). Suppose that the correction function $\mathcal{C}_{\varrho}^{h}(y-x,x)=1, \forall x \in \mathbb{R}^n$; $u \in H^{m+s+1}(\mathbb{R}^n)$; and the window function $\phi \in \mathcal{SF}_{1/a}^{(m+s+1)}$. The mth order MLSRK function can achieve a synchronized convergence rate for global interpolation, i.e.

$$\|u-\mathcal{R}_{\varrho,h}^{m}u\|_{H^k(\mathbf{R}^n)} \leqslant C_k \varrho^{m+1}\|u\|_{H^{k+m+1}(\mathbf{R}^n)} \quad 0 \leqslant k \leqslant s \tag{213}$$

REMARK 5.1. The assumption made in the theorem that $\phi \in \mathcal{SF}_{1/a}^{(m+s+1)}$ is just for convenience. One may easily show that the theorem holds, if only $\phi \in \mathcal{SF}_{1/a}^{(m+s)}$.

PROOF. The following proof is very similar to the proof of Young's inequality [1,31], except this is a semi-discrete version.

$$\begin{aligned} D_x^{\beta}\left(u-\mathcal{R}_{\varrho,h}^{m}u\right) &= D_x^{\beta}u(x) - \sum_{j\in\mathbf{Z}^n} u_j D_x^{\beta}\mathcal{K}_{\varrho}^{h}(x_j-x)h^n \\ &= D_x^{\beta}u(x) - \sum_{j\in\mathbf{Z}^n}\left\{\frac{1}{\alpha!}(x_j-x)^{\alpha}D_x^{\alpha}u(x)D_x^{\beta}\mathcal{K}_{\varrho}^{h}(x_j-x)h^n\right. \\ &\quad \left.+\sum_{|\alpha|=m+s+1}\frac{1}{\alpha!}(x_j-x)^{\alpha}D^{\alpha}u(x+\theta(x_j-x))D_x^{\beta}\mathcal{K}_{\varrho}^{h}(x_j-x)h^n\right\} \end{aligned} \tag{214}$$

where $|\beta| \leq s$, and $0<|\theta|<|\mathrm{h}|$.

By Lemma 5.2, we may have

$$-D_x^{\beta}\left(u-\mathcal{R}_{\varrho,h}^{m}u\right) = \sum_{j\in\mathbf{Z}^n}\sum_{|\alpha|=m+s+1}\frac{1}{\alpha!}(x_j-x)^{\alpha}D^{\alpha}u(x+\theta(x_j-x))D_x^{\beta}\mathcal{K}_{\varrho}^{h}(x_j-x)h^n \tag{215}$$

Let $t_j := x_j - x$. Then, Eq. (215) can be rewritten as

$$-D_x^{\beta}\left(u-\mathcal{R}_{\varrho,h}^{m}u\right) = \sum_{j\in\mathbf{Z}^n}\sum_{|\alpha|=m+s+1|}\frac{1}{\alpha!}D_x^{\alpha}u(x+\theta t_j)(t_j)^{\alpha}D_x^{\beta}\mathcal{K}_{\varrho}^{h}(t_j)h^n \tag{216}$$

Applying the Cauchy–Schwartz inequality to (216) and considering the fact that $|\alpha|=m+s+1$ is finite, one may find a constant, $0<C_0<\infty$, such that

$$\begin{aligned} |D_x^{\beta}(u-\mathcal{R}_{\varrho,h}^{m}u)|^2 \leqslant C_0 &\left\{\sum_{j\in\mathbf{Z}^n}\sum_{|\alpha|=m+s+1}\frac{1}{(\alpha!)}\sum_{j\in\mathbf{Z}^n}(t_j)^{\alpha}D_x^{\beta}\mathcal{K}_{\varrho}^{h}(t_j)h^n\right\} \\ &\cdot\left\{\sum_{j\in\mathbf{Z}^n}\sum_{|\alpha|=m+s+1}\frac{1}{(\alpha!)}(D_x^{\alpha}u(x+\theta t_j))^2(t_j)^{\alpha}D_x^{\beta}\mathcal{K}_{\varrho}^{h}(t_j)h^n\right\} \end{aligned} \tag{217}$$

By the Poisson summation formula (37) and the assumption $\phi \in \mathcal{SF}_{1/a}^{(2m+1)}$, one has

$$\sum_{j\in\mathbf{Z}^n}(t_j/\varrho)^{\alpha}D_x^{\beta}\mathcal{K}_{\varrho}^{h}(t_j)h^n = \sum_{j\in\mathbf{Z}^n}(i)^{|\alpha|-|\beta|}[D_{\zeta}^{\alpha}(\zeta^{\beta}\hat{\mathcal{K}}^{h}(\zeta))\exp(-\mathrm{i}\zeta x)]|_{\zeta=2\pi ja} \tag{218}$$

$$= (i)^{|\alpha|-|\beta|}\frac{\alpha!}{(\alpha-\beta)!}D_{\zeta}^{\alpha-\beta}\hat{\mathcal{K}}^{h}(0) \tag{219}$$

Substituting (219) into (217) yields

$$|D_x^\beta(u - \mathcal{R}_{\varrho,h}^m u)|^2 \leqslant C_0 \sum_{|\alpha|=m+s+1} \varrho^{m+1} \left\{ \frac{(i)^{|\alpha|-|\beta|}}{(\alpha-\beta)!} D_\zeta^{\alpha-\beta} \hat{\mathcal{K}}_\varrho^h(0) \right\}$$
$$\cdot \left\{ \sum_{j\in\mathbf{Z}^n} \sum_{|\alpha|=m+s+1} \frac{1}{\alpha!} \left(D_x^\alpha u(x+\theta t_j)\right)^2 (t_j)^\alpha D_x^\beta \mathcal{K}_\varrho^h(t_j) h^n \right\} \tag{220}$$

Thus,

$$\int_{\mathbf{R}^n} |D_x^\beta(u - \mathcal{R}_{\varrho,h}^m u)|^2 \,\mathrm{d}x \leqslant C_{\alpha,\beta} \varrho^{m+1} \sum_{|\alpha|=m+s+1} \frac{1}{\alpha!}$$
$$\cdot \left\{ \int_{\mathbf{R}^n} \sum_{j\in\mathbf{Z}^n} \left[\left(D_x^\alpha u(x+\theta t_j)\right)^2 (t_j)^\alpha \mathcal{K}_\varrho^h(t_j) h^n \right] \mathrm{d}x \right\} \tag{221}$$

For the sake of simplicity, without losing generality, one can assume that functions u, $\mathcal{K}_\varrho^h$ are smooth enough, such that the integral-summation in the left side of Eq. (221) can be bounded by a double integral form, i.e.

$$\left| \int_{\mathbf{R}^n} \sum_{j\in\mathbf{Z}^n} \left[\left(D_x^\alpha u(x+\theta t_j)\right)^2 (t_j)^\alpha D_x^\beta \mathcal{K}_\varrho^h(t_j) h^n \right] \mathrm{d}x \right|$$
$$\leqslant C_h \left| \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} \left[(D_x^\alpha u(x+\theta(y-x)))^2 (y-x)^\alpha D_x^\beta \mathcal{K}_\varrho^h(y-x) \right] \mathrm{d}y\,\mathrm{d}x \right| \tag{222}$$

Let $z = y - x$. Then,

$$\left| \int_{\mathbf{R}^n} \sum_{j\in\mathbf{Z}^n} \left[\left(D_x^\alpha u(x+\theta t_j)\right)^2 (t_j)^\alpha D_x^\beta \mathcal{K}_\varrho^h(t_j) h^n \right] \mathrm{d}x \right|$$
$$\leqslant C_h \left| \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} \left[(D_x^\alpha u(x+\theta z))^2 (z)^\alpha D_z^\beta \mathcal{K}_\varrho^h(z) \right] \mathrm{d}x\,\mathrm{d}z \right|$$
$$\leqslant C_h \varrho^{m+1} \left| \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} \left[(D_x^\alpha u(x+\theta z))^2 (z)^\alpha D_z^\beta \mathcal{K}(z) \right] \mathrm{d}z\,\mathrm{d}x \right|$$
$$\leqslant C_{h,\theta} \varrho^{m+1} \|D_x^\alpha u(x)\|_{L_2(\mathbf{R}^n)}^2 \left| \int_{\mathbf{R}^n} z^\alpha D_z^\beta \mathcal{K}^h(z)\,\mathrm{d}z \right|$$
$$= C_{h,\theta} \varrho^{m+1} \left| \frac{(i)^{\alpha+\beta}\alpha!}{(\alpha-\beta)!} D_\zeta^{\alpha-\beta} \hat{\mathcal{K}}^h(0) \right| \|D_x^\alpha u(x)\|_{L_2(\mathbf{R}^n)}^2 \tag{223}$$

Subsequently, one can find a constant, $C_{\alpha,\beta,h,\theta,\phi} > 0$, such that

$$\int_{\mathbf{R}^n} |D_x^\beta(u - \mathcal{R}_{\varrho,h}^m u)|^2 \,\mathrm{d}x \leqslant C_{\alpha,\beta,h,\theta,\phi} \varrho^{2m+2} \sum_{|\alpha|=m+s+1} \|D_x^\alpha u(x)\|_{L_2(\mathbf{R}^n)}^2 \tag{224}$$

Finally, we conclude that, $\exists\ 0 < C_k < \infty$, such that

$$\|u - \mathcal{R}_{\varrho,h}^m u\|_{H_k(\mathbf{R}^n)} \leqslant C_k \varrho^{m+1} \|u\|_{H_{k+m+1}(\mathbf{R}^n)}, \quad \forall 0 \leqslant k \leqslant s \quad \square \tag{225}$$

It should be noted that the synchronized convergence phenomenon has been observed in some other numerical methods as well, such as the numerical example reported by Carey and Oden in [6], which is calculated by using the Galerkin-collocation method.

5.3. Numerical examples

Two sets of numerical examples have been computed to verify the theoretical prediction. As mentioned above, for the kernel function with linear polynomial basis, all the functions in B-spline family whose order are greater than 3 can be a candidate window function, since they all satisfy the conditions:

$$\phi(\cdot) \in \mathcal{SF}^{(2)}\,, \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}\zeta}\phi(0) = 0\,, \tag{226}$$

in order that $C_\varrho^h(y - x, x) = 1$ in the interior region of a specific domain. For instance, for the cubic spline, $\phi_3 \in \mathcal{SF}^{(3)}$, and for the fifth-order B-spline, $\phi_5 \in \mathcal{SF}^{(5)}$, and so on.

By choosing both the cubic spline function and the fifth order spline function as the window function, numerical calculations are carried out in solving the following Neumann problem in a finite domain.

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2}u + u = f(x) \quad \forall x \in (-1, 1) \tag{227}$$

$$f(x) = x^6 - 30.0x^4\,, \qquad \left.\frac{\mathrm{d}u}{\mathrm{d}x}\right|_{x=-1} = -6, \qquad \left.\frac{\mathrm{d}u}{\mathrm{d}x}\right|_{x=1} = 6\,. \tag{228}$$

In the numerical computation, the particle number varies from 11 to 200. Some convergence results are tabulated as well as plotted in both Tables 1, 2 and Figs. 3 and 4. In Tables 1 and 2, the error norms are compared with the length of the compact support of the window function, which is proportional to the dilation parameter, i.e. $R = n\varrho$. In these examples, $n = 2$ for cubic spline window function, and $n = 3$ for the fifth-order splines window function, though they may not be important to the final results of the convergence rates at all.

From Figs. 3 and 4, one can observe a strikingly prominent 'synchronized effect'—all the error norms have the same convergence rate!

In numerical example I, $m = 1, s = 2, (m + s = 3)$; Theorem 5.2 guarantees the effect of synchronized convergence up to H_2 error norm. Similarly, in numerical example II, $m = 1, s = 4, (m + s = 5)$; Theorem

Table 1
Convergence results by using cubic spline with $C_\varrho = 1$ and $n = 2$

$\log(n\varrho)$	$\log(\lVert\text{error}\rVert_{L_2})$	$\log(\lVert\text{error}\rVert_{H_1})$	$\log(\lVert\text{error}\rVert_{H_2})$
−0.3688879454E+01	−0.1136431870E+02	−0.1035178324E+02	−0.5262474429E+01
−0.3912023005E+01	−0.1169823788E+02	−0.1135594391E+02	−0.6199262484E+01
−0.4094344562E+01	−0.1199565056E+02	−0.1179989480E+02	−0.6676681866E+01
−0.4248495242E+01	−0.1225887402E+02	−0.1210113021E+02	−0.6990833887E+01
−0.4382026635E+01	−0.1249353413E+02	−0.1235519383E+02	−0.7273819430E+01
−0.4499809670E+01	−0.1270465718E+02	−0.1258149130E+02	−0.7503394557E+01
−0.4605170186E+01	−0.1289628599E+02	−0.1278076822E+02	−0.7675119133E+01

Table 2
Convergence results obtained by using fifth-order B-spline with $C_\varrho = 1$ and $n = 3$

$\log(n\varrho)$	$\log(\lVert\text{error}\rVert_{L_2})$	$\log(\lVert\text{error}\rVert_{H_1})$	$\log(\lVert\text{error}\rVert_{H_2})$	$\log(\lVert\text{error}\rVert_{H_3})$
−0.4199705E+01	−0.1516584E+02	−0.1509792E+02	−0.1476808E+02	−0.1098707E+02
−0.4295015E+01	−0.1544146E+02	−0.1537354E+02	−0.1506008E+02	−0.1224574E+02
−0.4382027E+01	−0.1569229E+02	−0.1562437E+02	−0.1531028E+02	−0.12747731E+02
−0.4462069E+01	−0.1592232E+02	−0.1585440E+02	−0.1554048E+02	−0.13022686E+02
−0.4536177E+01	−0.1613464E+02	−0.1606673E+02	−0.1575287E+02	−0.13244296E+02
−0.4605170E+01	−0.1633172E+02	−0.1626380E+02	−0.1594997E+02	−0.13451707E+02

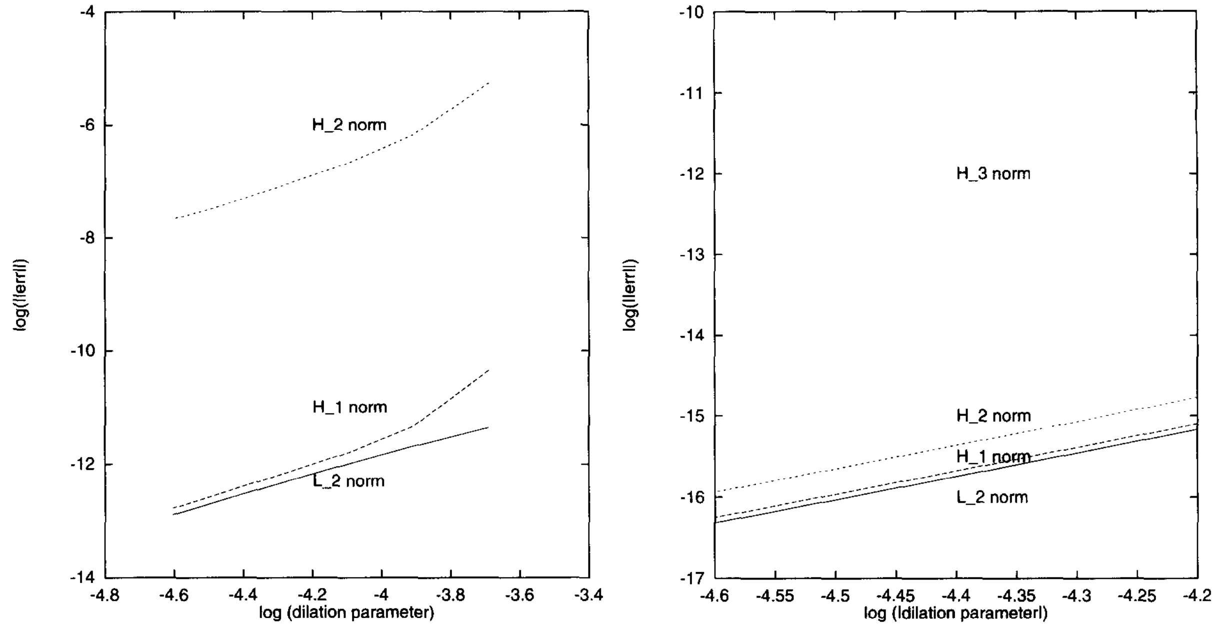

Fig. 3. Synchronized convergence case I: (cubic spline).

Fig. 4. Synchronized convergence case II: (fifth-order B-spline).

5.2 actually guarantees the synchronized convergence up to H_4 error norm. Nevertheless, Fig. 4 only displays numerical results up to H_3 error norm.

6. Closure

The development of technology as a human endeavor is hardly aiming at a higher moral cause, but simply to make life easier. While judging whether this or that technology has the enduring quality or not, we are often in favor of that, which will liberate us from the tedious, monotonous, physical labor. In this sense, the meshless method is a winner at the philosophical level at least.

At this point, some people might feel that the meshless method is a secondary improvement or complementary revision over the conventional finite element method. From our perspective, it may be more than that. There are some fundamental changes in the ways that we used to look at the problems. For instance, here, the basic premise of p-finite element, in which accuracy is enhanced only by increase the order of interpolation polynomials, is challenged, because, as shown in the paper, one may achieve an increase of accuracy by only tuning the particle distribution from an arbitrary distribution to a uniform distribution, such that the order of regularity of the kernel function can increase drastically, which subsequently leads an increase in the approximation order of the numerical computation.

Once again, the reader should be reminded that the results presented in this paper are only valid for uniform meshes, or uniform particle distributions. Nevertheless, as pointed out earlier, these results can serve as an interior estimate, and this is actually the usual way that is performed in practice. The same argument can be applied to the synchronized convergence phenomenon, since in practice the correction function is usually at order 1, i.e.

$$\mathcal{C}^h_\varrho \sim \mathcal{O}(1) \tag{229}$$

A subsequent question is: would the synchronized convergence effect vanish, if $\mathcal{C}^h_\varrho \neq 1$, or how far would general case deviate from the ideal case. To answer this question, one might need a further in-depth

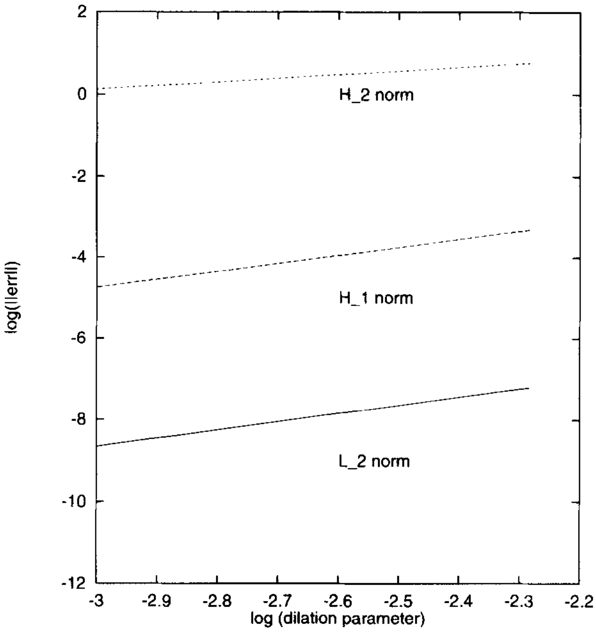

Fig. 5. Synchronized convergence case III: ($\mathcal{C}_\varrho \neq 1$).

study. Here, without much involvement and further complication, we only show a convergence result of a numerical experiment in Fig. 5, in which the correction function is not constant, i.e. $\mathcal{C}^h_\varrho(y - x, x) \neq 1$. One may find that there is still a strong background 'synchronized effect' for the H_1 error norm as anticipated.

Some readers may wonder that since the window function has a higher-order approximation power, why do we not use it as the kernel function in the first place, rather going through all these troubles instead. The reason are again two-fold: first, if one uses the window function as the kernel function, most likely, the numerical algorithm won't be a meshless method. Second, it may also not satisfy the boundary condition generally, whereas MLSR kernel function remains its meshless status and at the same time enjoys a background high power convergence rate.

Acknowledgment

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Appendix A. Moving least-square reproducing kernel formula

For the sake of self-contained presentation, we append some basic MLSRK formulas here, which are taken from Part I of this work [21].

Let $u(x)$ be a sufficiently smooth function, which is defined on a simply connected open set $\Omega \in \mathbb{R}^n$. Thus, by the Stone–Weierstrass theorem, we can define a local function as follows, $\forall \bar{x} \in \Omega$,

$$u^l(x, \bar{x}) := \begin{cases} u(x), & \forall x \in \boldsymbol{B}(\bar{x}) \\ 0, & \text{otherwise} \end{cases} \tag{A.1}$$

where the open sphere $\boldsymbol{B}(\bar{x})$ is defined as

$$\boldsymbol{B}(\bar{x}) := \{x : |x - \bar{x}| < \epsilon, x \in \bar{\Omega}\} \tag{A.2}$$

If the function $u(x)$ is smooth enough as assumed, there exists a local operator, such that

$$u^l(x,\bar{x}) \cong L_{\bar{x}}u(x) := \sum_{i=1}^{\ell} P_i\left(\frac{x-\bar{x}}{\varrho}\right) d_i(\bar{x}) = \boldsymbol{P}\left(\frac{x-\bar{x}}{\varrho}\right)\boldsymbol{d}(\bar{x}) \tag{A.3}$$

The operator $L_{\bar{x}}$ is a mapping

$$L_{\bar{x}} : C^0(\boldsymbol{B}(\bar{x})) \longmapsto C^m(\boldsymbol{B}(\bar{x})) \tag{A.4}$$

which is characterized by its basis,

$$\boldsymbol{P}(x) := \{P_1, P_2, \dots, P_\ell\}(x) \tag{A.5}$$

where

$$P_1(x) := 1; \tag{A.6}$$

$$P_i(x) := \left(\frac{x-\bar{x}}{\varrho}\right)^{\alpha_{i-1}}, \quad 2 \leqslant i \leqslant \ell\ , \ |\alpha_i| \leqslant m \tag{A.7}$$

and

$$\boldsymbol{d}^t(x) := \{d_1, d_2, \dots, d_\ell\}(x). \tag{A.8}$$

Since the polynomial expansion is a finite sum, there exists an error residual distribution r_ϱ over the ball $\boldsymbol{B}(\bar{x})$

$$r_\varrho(x,\bar{x}) := u^l(x) - \boldsymbol{P}\left(\frac{x-\bar{x}}{\varrho}\right)\boldsymbol{d}(\bar{x}), \quad x \in \boldsymbol{B}(\bar{x}) \tag{A.9}$$

A functional associated with this residual is defined as

$$J(\boldsymbol{d}(\bar{x})) := \int_{\boldsymbol{B}(\bar{x})} \frac{1}{\varrho^n} r_\varrho^2(x,\bar{x})\phi\left(\frac{x-\bar{x}}{\varrho}\right) \mathrm{d}\boldsymbol{B} \tag{A.10}$$

The weight function inside the integrand satisfies the requirements

$$\phi(\cdot) \in \mathcal{S}_r \cap C_c^m(\Omega) \quad r >> m\ , \quad \text{and} \quad \int_\Omega \phi\, \mathrm{d}\Omega = 1 \tag{A.11}$$

And since ϕ is compact supported, $\exists B(\varrho,\bar{x}) \subset \boldsymbol{B}(\bar{x})$, such that

$$\phi_\varrho(x-\bar{x}) := \frac{1}{\varrho^n}\phi\left(\frac{x-\bar{x}}{\varrho}\right) := \begin{cases} >0, & x \in B(\varrho,\bar{x}) \\ 0, & x \notin B(\varrho,\bar{x}) \end{cases} \tag{A.12}$$

By minimizing the quadratic functional $J(\boldsymbol{d}(\bar{x}))$, one may obtain the following normal equation

$$\int_{\boldsymbol{B}(\bar{x})} \boldsymbol{P}^t\left(\frac{x-\bar{x}}{\varrho}\right)\left(u^l(x) - \boldsymbol{P}\left(\frac{x-\bar{x}}{\varrho}\right)\boldsymbol{d}(\bar{x})\right)\phi_\varrho(x-\bar{x})\,\mathrm{d}\boldsymbol{B} = 0 \tag{A.13}$$

Since $\operatorname{supp}\{\phi_\varrho(x-\bar{x})\} \subseteq \bar{\boldsymbol{B}}$, Eq. (A.13) leads to

$$\left(\int_{\Omega_x} \boldsymbol{P}^t\left(\frac{x-\bar{x}}{\varrho}\right)\phi_\varrho(x-\bar{x})\boldsymbol{P}\left(\frac{x-\bar{x}}{\varrho}\right)\mathrm{d}\Omega_x\right)\boldsymbol{d}(\bar{x}) = \int_{\Omega_x} \boldsymbol{P}^t\left(\frac{x-\bar{x}}{\varrho}\right)u(x)\phi_\varrho(x-\bar{x})\,\mathrm{d}\Omega_x \tag{A.14}$$

To this end, one can define the local moment matrix $\boldsymbol{M}_\varrho(\bar{x})$ as follows

$$\boldsymbol{M}_\varrho(\bar{x}) := \int_\Omega \boldsymbol{P}^t\left(\frac{x-\bar{x}}{\varrho}\right)\phi_\varrho(x-\bar{x})\boldsymbol{P}\left(\frac{x-\bar{x}}{\varrho}\right)\mathrm{d}\Omega \tag{A.15}$$

or simply

$$M_\varrho(\bar{x}) := \langle \boldsymbol{P}^t|\phi_\varrho|\boldsymbol{P}\rangle_x = \begin{pmatrix} \langle P_1|\phi_\varrho|P_1\rangle & \langle P_1|\phi_\varrho|P_2\rangle & \cdots & \langle P_1|\phi_\varrho|P_m\rangle \\ \langle P_2|\phi_\varrho|P_1\rangle & \langle P_2|\phi_\varrho|P_2\rangle & \cdots & \langle P_2|\phi_\varrho|P_m\rangle \\ \vdots & & \ddots & \vdots \\ \langle P_m|\phi_\varrho|P_1\rangle & \langle P_m|\phi_\varrho|P_2\rangle & \cdots & \langle P_m|\phi_\varrho|P_m\rangle \end{pmatrix}. \tag{A.16}$$

where the inner product operation $\langle |,| \rangle$ is defined as

$$\langle f|\phi_\varrho|g\rangle_y := \int_{\Omega_y} f\left(\frac{y-x}{\varrho}\right) g\left(\frac{y-x}{\varrho}\right) \phi_\varrho(x-y)\,\mathrm{d}\Omega_y \tag{A.17}$$

By inverting $\boldsymbol{M}_\varrho(\bar{x})$, the unknown vector $\boldsymbol{d}(\bar{x})$ is uniquely determined as,

$$\boldsymbol{d}(\bar{x}) = \boldsymbol{M}_\varrho^{-1}(\bar{x}) \int_{\Omega_x} \boldsymbol{P}^t\left(\frac{x-\bar{x}}{\varrho}\right) u(x)\phi_\varrho(x-\bar{x})\,\mathrm{d}\Omega_x \tag{A.18}$$

Substituting (A.18) back into (A.3) and noting that the argument x in (A.18) is a dummy variable, one can derive a local reproducing kernel formula as

$$\begin{aligned} u^l(x,\bar{x}) \cong L_{\bar{x}}u(x) &:= \boldsymbol{P}\left(\frac{x-\bar{x}}{\varrho}\right)\boldsymbol{d}(\bar{x}) \\ &= \boldsymbol{P}\left(\frac{x-\bar{x}}{\varrho}\right)\boldsymbol{M}_\varrho^{-1}(\bar{x}) \int_{\Omega_y} \boldsymbol{P}^t\left(\frac{y-\bar{x}}{\varrho}\right) u(y)\phi_\varrho(y-\bar{x})\,\mathrm{d}\Omega_y \end{aligned} \tag{A.19}$$

Note that here Eq. (A.19) is only a local approximation: $\forall x \in \boldsymbol{B}(\bar{x})$. In order to obtain a global approximation, we introduce a global approximation operator G,

$$u(x) \cong Gu(x), \quad \forall x \in \bar{\Omega} \tag{A.20}$$

where

$$G : C^0(\bar{\Omega}) \longmapsto C^m(\bar{\Omega}) \tag{A.21}$$

More precisely, the global approximation operator G is defined in such a way that it is the globalization of the local approximate operator $L_{\bar{x}}$, and the globalization is realized through the moving process, viz.

$$Gu(x) := \lim_{\bar{x}\to x} L_{\bar{x}}u(x), \quad \forall x \in \bar{\Omega} \tag{A.22}$$

At the end of the sweeping process, one can get $Gu(x) := L_x u(x)$, viz.

$$Gu(x) = \boldsymbol{P}(0)\boldsymbol{M}_\varrho^{-1}(x) \int_{\Omega_y} \boldsymbol{P}^t\left(\frac{y-x}{\varrho}\right) u(y)\phi_\varrho(y-x)\,\mathrm{d}\Omega_y \tag{A.23}$$

Eq. (A.23) is the so-called '*moving least-square reproducing kernel*' formulation, which is named by Liu et al. [20].

The components of the moment matrix can be defined as follows:

$$M_{ij} := \int_\Omega P_i(x)P_j(x)\phi(x)\,\mathrm{d}\Omega \tag{A.24}$$

$$= \int_\Omega x^{(\alpha_{i-1}+\alpha_{j-1})}\phi(x)\,\mathrm{d}\Omega, \quad i,j = 1,\ldots,\ell \tag{A.25}$$

where $D_\ell := \det|\boldsymbol{M}_\varrho|$, and A_{ij} are the minors of the momnet matrix $\boldsymbol{M}_\varrho$.

Eq. (A.23) can then be rewritten in a succinct manner,

$$\begin{aligned} Gu(x) &:= \boldsymbol{P}(0)\boldsymbol{M}_\varrho^{-1}(x) \int_{\Omega_y} \boldsymbol{P}^t\left(\frac{y-x}{\varrho}\right) u(y)\phi_\varrho(y-x)\,\mathrm{d}\Omega_y \\ &= \int_{\Omega_y} C_\varrho(y-x,x)u(y)\phi_\varrho(y-x)\,\mathrm{d}\Omega_y \end{aligned} \tag{A.26}$$

where function $C_\varrho(y-x,x)$ is the so-called *correction function* [20], which is defined as follows:

$$\mathcal{C}_\varrho(y-x,x) := \boldsymbol{P}(0)\boldsymbol{M}_\varrho^{-1}(x)\boldsymbol{P}^t\left(\frac{y-x}{\varrho}\right) \tag{A.27}$$

The expression for the correction function can be further simplified as follows

$$\begin{aligned}
\mathcal{C}_\varrho(y-x,x) &= \boldsymbol{P}(0)\boldsymbol{M}_\varrho^{-1}(x)\boldsymbol{P}^t\left(\frac{y-x}{\varrho}\right)\\
&= (1,0,0,\dots,0)\,\frac{1}{D_\ell}\\
&\begin{pmatrix} A_{11} & -A_{12} & \cdots & (-1)^{1+i}A_{1i} & \cdots & (-1)^{1+\ell}A_{1\ell}\\ -A_{21} & A_{22} & \cdots & \cdots & \cdots & (-1)^{2+\ell}A_{2\ell}\\ \vdots & & \ddots & & & \vdots\\ (-1)^{j+1}A_{j1} & & & (-1)^{i+j}A_{ji} & & (-1)^{j+\ell}A_{j\ell}\\ \vdots & & & & \ddots & \vdots\\ (-1)^{\ell+1}A_{\ell 1} & & \cdots & (-1)^{\ell+i}A_{\ell i} & \cdots & A_{\ell\ell}\end{pmatrix}\begin{pmatrix}P_1\\P_2\\ \vdots\\ P_j\\ \vdots\\ P_\ell\end{pmatrix}\\
&= \frac{1}{D_\ell}[A_{11}P_1 - A_{12}P_2\dots,+(-1)^{1+i}A_{1i}P_i\dots,+(-1)^{1+\ell}A_{1\ell}P_\ell]\\
&= \boldsymbol{P}\left(\frac{y-x}{\varrho}\right)\boldsymbol{b}
\end{aligned} \tag{A.28}$$

where the vector $\boldsymbol{b}$ is defined as

$$\boldsymbol{b}^t(x) := \frac{1}{D_\ell(x)}[A_{11}(x), -A_{12}(x), A_{13}(x),\dots,(-1)^{1+\ell}A_{1\ell}(x)]. \tag{A.29}$$

Let

$$\mathcal{K}_\varrho(y-x,x) := \mathcal{C}_\varrho(y-x,x)\phi_\varrho(y-x). \tag{A.30}$$

The moving least-square reproducing kernel integral representation takes the form

$$\mathcal{R}_\varrho^m u(x) := \int_\Omega u(y)\mathcal{K}_\varrho(x-y,x)\,\mathrm{d}\Omega \tag{A.31}$$

here the supscript m represents the highest order of the generating polynomial. It should be noted that since the moments of the window function are constants for infinitive domain, e.g. $\mathbb{R}^n$, both kernel function and correction function can be simplified as one argument functions, i.e.

$$\mathcal{K}_\varrho(y-x,x) = \mathcal{K}_\varrho(y-x), \qquad \mathcal{C}_\varrho(y-x,x) = \mathcal{C}_\varrho(y-x)\ , \tag{A.32}$$

whereas the discrete kernel function as well as discrete correction function are always the functions with two arguments. Accordingly, the discrete MLSRK formula is defined as follows:

$$\mathcal{R}_{\varrho,h}^m u(x) = \sum_{i=1}^{np} u(x_i)\mathcal{K}_\varrho^h(x_i-x,x)\,\Delta V_i = \sum_{i=1}^{np} u(x_i)\mathcal{C}_\varrho^h(x_i-x,x)\phi_\varrho(x_i-x)\,\Delta V_i \tag{A.33}$$

with the compatible discrete moment matrix

$$\boldsymbol{M}_\varrho^h(x) = \left\{M_{ij}^h\right\} \tag{A.34}$$

$$M_{ij}^h(x) := \sum_{k=1}^{np}\left(\frac{x_k-x}{\varrho}\right)^{\alpha_{i-1}+\alpha_{j-1}}\phi_\varrho(x_k-x)\,\Delta V_k \tag{A.35}$$

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