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On Dual Conservation Laws in Linear Elasticity: Stress Function Formalism

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Abstract. Dual conservation laws of linear planar elasticity theory have been systematically studied based on stress function formalism. By employing generalized symmetry transformation or the Lie–Bäcklund transformation, a class of new dual conservation laws in planar elasticity have been discovered based on the Noether theorem and its Bessel–Hagen generalization. The physical implications of these dual conservation laws are discussed briefly.

Key words: conservation laws, elasticity, J-integral, Lie group, Lie–Bäcklund transformation

1. Introduction

The conservation law of elasticity has been well studied in the past 30 years, though the origin of its intellectual inspiration may be traced back to Eshelby's seminal work in 1950s [1, 2]. Since Rice [3] linked J-integral with the energy release rate of a crack, the subject has become part of the theoretical foundation of fracture mechanics. Contributions on conservation laws of elasticity include: [4–10, 27, 28], among others.

Classical elasticity is a perfect embodiment of duality, in which strain representation and stress representation complement each other to describe a complete image of an equilibrium-deformation process. On the variational level, the duet are the minimal potential energy principle and the minimal complementary energy principle. Since conservation laws of elasticity are manifestation of symmetry properties of variational principles in elasticity, naturally, conservation laws of elasticity ought to come as dual pairs, and they should be displayed on an equal footing. Indeed, some authors have studied conservation laws based on the complementary variational principle. Several dual invariant integrals or dual conservation laws have been derived. Among them, the dual J-integral derived by Bui [11] is the earliest contribution. Other notable contributions include those of Sun [12] and Li [13].

The early studies on dual conservation laws are mainly based on physical observation or intuition via direct divergence-free inspection. The path integrals derived are indeed invariant. However, most early studies are not only incomplete, they also do not match the standard of elegance and rigor that are usually expected in continuum mechanics. Part of the reason may be attributed to lack of serious attention to the subject. Probably the lack of proper physical interpretation of dual conservation laws can be attributed as another reason for such public oblivion. In fact, most dual conservation laws published in the literature are trivial in the sense that they can be easily obtained by integration by parts from the conservation laws of the potential energy variational principle, which may give an impression that the conservation laws of the Navier equations may have exhausted all the possible symmetry properties of linear elasticity. The dual conservation laws may be just a repetition of the conservation law derived from the minimal potential energy principle, and no more non-trivial conservation laws are left undiscovered in linear elasticity theory.

Apparently, this is a false impression. Over the years, people have realized that additional conservation laws may still exist in linear elasticity, and some of them remain undiscovered. The conservation laws discovered by Christiansen et al. [14, 15] and the conservation laws derived independently by Horgan et al. and Flavin [16–19] are such examples. In fact, these conservation laws are very useful as theoretical apparatuses in applications, e.g. estimating energy bounds, justifying the Saint-Venant principle, and possibly in convergence study of finite element methods.

2. Preliminary

2.1. Planar Elasticity

To fix the notation, we start by reviewing some basic facts of planar elasticity. For a plane stress state, linear two-dimensional (2D) elastic constitutive relations can be written,

$$\varepsilon_{\alpha\beta} = C_{\alpha\beta\lambda\mu}\sigma_{\lambda\mu} \quad \text{or} \quad \sigma_{\alpha\beta} = E_{\alpha\beta\lambda\mu}\varepsilon_{\lambda\mu} \tag{1}$$

where $\varepsilon_{\alpha\beta}$, $\sigma_{\alpha\beta}$ are the usual strain, stress tensor, respectively; and $C_{\alpha\beta\lambda\mu}$, $E_{\alpha\beta\lambda\mu}$ are the elastic compliance, and stiffness tensor respectively. Note that the Einstein summation convention is implicitly assumed throughout the paper.

For isotropic, homogeneous elastic materials, the 2D elastic compliance tensor can be expressed as

$$C_{\alpha\beta\lambda\mu} := \frac{1+\nu}{E}\delta_{\alpha\lambda}\delta_{\beta\mu} - \frac{\nu}{E}\delta_{\alpha\beta}\delta_{\lambda\mu}; \tag{2}$$

and the 2D elastic stiffness modulus tensor has the form

$$E_{\alpha\beta\lambda\mu} := \frac{E}{1+\nu}\delta_{\alpha\lambda}\delta_{\beta\mu} + \frac{E\nu}{1-\nu^2}\delta_{\alpha\beta}\delta_{\lambda\mu}, \tag{3}$$

where E is Young's modulus, and ν is the Poisson ratio.

The above elastic stiffness tensor and compliance tensor are only valid in the plane stress state. To find the elastic stiffnes tensor and compliance tensor in the plane strain state, one can replace Young's modulus and Poisson's rate by

$$E \Rightarrow \frac{E}{1-\nu^2}, \qquad \nu \Rightarrow \frac{\nu}{1-\nu}, \tag{4}$$

and the corresponding tensors in the plane strain state are:

$$C_{\alpha\beta\lambda\mu} = \frac{1+\nu}{E}\delta_{\alpha\lambda}\delta_{\beta\mu} - \frac{(1+\nu)\nu}{E}\delta_{\alpha\beta}\delta_{\lambda\mu}, \tag{5}$$

$$E_{\alpha\beta\lambda\mu} = \frac{E}{1+\nu}\delta_{\alpha\lambda}\delta_{\beta\mu} + \frac{E\nu}{(1+\nu)(1-2\nu)}\delta_{\alpha\beta}\delta_{\lambda\mu}. \tag{6}$$

In the rest of paper, we mainly deal with the plane stress description, with the understanding that all the results are valid for the plane strain description as well, unless it is indicated otherwise.

In absence of body force, one may introduce the Airy stress function, such that

$$\sigma_{\alpha\beta} = \epsilon_{\alpha\lambda}\epsilon_{\beta\mu}\varphi_{,\lambda\mu} \quad \text{or} \quad \varphi_{,\alpha\beta} = \epsilon_{\alpha\lambda}\epsilon_{\beta\mu}\sigma_{\lambda\mu}, \tag{7}$$

where $\epsilon_{\alpha\beta}$ is the 2D permutation tensor, i.e.

$$[\epsilon_{\alpha\beta}] = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{8}$$

A useful identity of the 2D permutation tensor is self-contraction,

$$\epsilon_{\gamma\alpha}\epsilon_{\gamma\beta} = \delta_{\alpha\beta}. \tag{9}$$

For the plane stress state, the density of the complementary energy is

$$L_c(\boldsymbol{\sigma}; \mathbf{x}) = \frac{1}{2E}[(1+\nu)\sigma_{\alpha\beta}\sigma_{\alpha\beta} - \nu\sigma_{\lambda\lambda}\sigma_{\mu\mu}] \tag{10}$$

In terms of the Airy stress function, the complementary energy density in the plane stress state can be written as

$$L_c^{(se)}(\partial^2\varphi; \mathbf{x}) = \frac{1}{2E}[(1+\nu)\varphi_{,\alpha\beta}\varphi_{,\alpha\beta} - \nu\varphi_{,\lambda\lambda}\varphi_{,\mu\mu}] \tag{11}$$

where $\partial^2\varphi = \{\varphi_{,\alpha\beta}\}$, $\alpha, \beta = 1, 2$ and the complementary energy density in the plane strain state is

$$L_c^{(sa)}(\partial^2\varphi; \mathbf{x}) = \frac{(1+\nu)}{2E}[\varphi_{,\alpha\beta}\varphi_{,\alpha\beta} - \nu\varphi_{,\lambda\lambda}\varphi_{,\mu\mu}] \tag{12}$$

Assume that the stress function is prescribed over the whole boundary. The total complementary potential energy is

$$\Pi_c(\varphi; \mathbf{x}) = \iint_\Omega L_c(\partial^2\varphi; \mathbf{x})d\Omega. \tag{13}$$

The Euler–Lagrangian equation of the complementary energy functional is the biharmonic equation, i.e.

$$\frac{\partial L_c}{\partial\varphi} - \frac{\partial}{\partial x_\alpha}\frac{\partial L_c}{\partial\varphi_{,\alpha}} + \frac{\partial^2}{\partial x_\alpha \partial x_\beta}\frac{\partial L_c}{\partial\varphi_{,\alpha\beta}} = 0 \Rightarrow \varphi_{,\alpha\alpha\beta\beta} = 0, \tag{14}$$

which carries different information from the 2D Navier equations. Specifically, the biharmonic equation satisfied by stress function, φ, characterizes the compatibility constraint of 2D elasticity.

Given linear operator $\mathbf{L} = L_{\alpha\beta\lambda\mu}\mathbf{e}_\alpha \otimes \mathbf{e}_\beta \otimes \mathbf{e}_\lambda \otimes \mathbf{e}_\mu$; and $\mathbf{a} = a_\alpha\mathbf{e}_\alpha \neq 0$, $\mathbf{b} = b_\alpha\mathbf{e}_\alpha \neq 0$; $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2$. Let $\mathbf{A} := \mathbf{a} \otimes \mathbf{b}$. We say that $\mathbf{L}$ is strongly elliptic if

$$\mathbf{A{:}L{:}A} = L_{\alpha\beta\lambda\mu}a_\alpha b_\beta a_\lambda b_\mu > 0\,. \tag{15}$$

It is not difficult to verify the positive definite condition for complementary potential energy,

$$\frac{1+\nu}{E} > 0, \qquad \frac{1-2\nu}{E} > 0 \qquad \text{(plane stress);} \tag{16}$$

$$\frac{1+\nu}{E} > 0, \qquad \frac{(1+\nu)(1-2\nu)}{E} > 0 \qquad \text{(plane strain).} \tag{17}$$

Therefore, for a two-dimensional elastic solid, a positive definite elastic compliance tensor is equivalent to $E > 0$ and $-1 < \nu < 1/2$, which is the same as in three-dimensional elastic solids (see [20]).

2.2. One-Parameter Group of Invariant Transformation

The Lie group analysis of partial differential equations has been a triumph in mathematics, physics, and engineering science. For contemporary expositions, readers may consult monographs by Ibragimov, and Olver [21–23]. In this section, we briefly summarize the main technical ingredients of generalized symmetry transformation, or the Lie–Bäcklund transformation, and Noether's theorem [29]. The notation adopted in this paper mainly follows Olver [23] and Ibragimov [22]. In this paper, we are only interested in the partial differential equation of a scalar function and its associated variational problem.

Let

$$\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n, \tag{18}$$

where $\mathbb{R}^n$ is the n-dimensional Cartesian space and

$$\partial u = \left\{ \frac{\partial u}{\partial x_i} \right\}, \dots\dots, \qquad \partial^s u = \left\{ \frac{\partial^s u}{\partial x_i^s} \right\}, \dots, \quad 1 < s, \quad 1 \le i \le n. \tag{19}$$

The space $\mathbf{Z}$ is a direct product,

$$\mathbf{Z} = \mathbb{R}^n \times V, \tag{20}$$

where V is an infinite dimensional vector space with component

$$\mathbf{y} = (u, \partial u, \dots, \partial^s u, \dots) \in V. \tag{21}$$

The point $\mathbf{z} = (z_1, z_2, \dots) \in \mathbf{Z}$ can be written as

$$\mathbf{z} = (\mathbf{x}, \mathbf{y}) = (x_1, x_2, \dots, x_n, u, \partial u, \dots, \partial^s u, \dots). \tag{22}$$

Denote the vector space of all differential functions of finite order m as $\mathcal{A}$ and any finite sequence of $\mathbf{z}$ as $[\mathbf{z}]$. Then elements of $\mathcal{A}$ may be written as $f([\mathbf{z}]) \in \mathcal{A}$.

Consider a formal one-parameter group G of generalized transformation of the following type,

$$x_i^* = \exp(\epsilon \xi_i) x_i; \tag{23}$$

$$u^* = \exp(\epsilon \eta) u; \tag{24}$$

where

$$\xi_i := \left. \frac{dx_i^*}{d\epsilon} \right|_{\epsilon=0}, \tag{25}$$

$$\eta := \left. \frac{du^*}{d\epsilon} \right|_{\epsilon=0}. \tag{26}$$

The transformation is generalized in a sense that its infinitesimal generators have the form

$$\xi_i = \xi_i(\mathbf{x}, u, \partial u, \dots, \partial^s u, \dots) = \xi_i(\mathbf{z}), \quad 1 \le i \le n \tag{27}$$

$$\eta = \eta(\mathbf{x}, u, \partial u, \dots, \partial^s u, \dots) = \eta(\mathbf{z}). \tag{28}$$

Definition 2.1 (Olver [23]). A generalized vector field is a formal expression of the following form

$$\mathbf{v} = \xi_i(\mathbf{z})\frac{\partial}{\partial x_i} + \eta(\mathbf{z})\frac{\partial}{\partial u} \tag{29}$$

Themorem 2.1 (prolongation formula) (Ibragimov [21]*, Olver* [23]*).* *An infinite prolongation (or prolongation for short) formula of the generalized (Lie–Bäcklund) vector field* **v** *is*

$$\mathrm{pr}\mathbf{v} = \mathbf{v} + \sum_{1\le s} \eta^{(s)}_{i_1\cdots i_s}\frac{\partial}{\partial u_{i_1\cdots i_s}} \tag{30}$$

where

$$\eta^{(s)}_{i_1\cdots i_s} = D_{i_1}\cdots D_{i_s}(\eta - \xi_j u_{,j}) + \xi_j u_{,ji_1\cdots 1_s}, \quad s = 1, 2, \ldots \tag{31}$$

and

$$D_i := \frac{\partial}{\partial x_i} + u_{,i}\frac{\partial}{\partial u} + \cdots + u_{,ii_1\cdots i_{s-1}}\frac{\partial}{\partial u_{,i_1\cdots,i_s}} + \cdots. \tag{32}$$

A p-th order prolongation formula of generalized vector field **v** *is*

$$\mathrm{pr}^{(p)}\mathbf{v} = \mathbf{v} + \sum_{1\le s\le p} \eta^{(s)}_{i_1\cdots i_s}\frac{\partial}{\partial u_{i_1\cdots i_s}} \tag{33}$$

where p is the maximal order of non-vanishing derivatives, and

$$\eta^{(s)}_{i_1\cdots i_s} = D_{i_1}\cdots D_{i_s}(\eta - \xi_j u_{,j}) + \xi_j u_{,ji_1\cdots 1_s}, \quad 1 \le s \le p \tag{34}$$

$$D_i = \frac{\partial}{\partial x_i} + u_{,i}\frac{\partial}{\partial u} + \cdots + u_{,ii_1\cdots i_{p-1}}\frac{\partial}{\partial u_{,i_1\cdots,i_p}}. \tag{35}$$

Consider a q-th order scalar partial differential equation (PDE) denoted by

$$F = F([\mathbf{z}]) = F(\mathbf{x}, u, \partial u, \ldots, \partial^q u), \tag{36}$$

where $q \ge 1$ is some positive integer. Define the differential manifold

$$[F] : F = 0, \quad \ldots \quad D_{1_1}\ldots D_{i_k}F = 0, \quad k = p + q. \tag{37}$$

We have the following theorem.

Theorem 2.2 *(Ibragimov* [21]*, Olver* [23]*).* *Let G be a group of the Lie–Bäcklund transformation, with tangent vector field* **prv**. *The differential manifold* $[F]$ *is invariant under G, if and only if*

$$\mathrm{pr}\mathbf{v}F\,\Big|_{[F]} = 0. \tag{38}$$

Note that

$$\mathrm{pr}\mathbf{v}F = 0 \ \Rightarrow \mathrm{pr}^{(q)}\mathbf{v}F([\mathbf{z}]) = 0. \tag{39}$$

Equation (39) is often referred to as the determining equation.

Define the Euler–Lagrangian operator

$$E := \frac{\partial}{\partial u} + \sum_{1\le s}(-1)^s D_{1_1} \ldots D_{i_s} \frac{\partial}{\partial u_{i_1\cdots i_s}}, \tag{40}$$

and the Noether operator

$$N_i = \xi_i + (\eta - \xi_\ell u_{,\ell}) \left\{ \frac{\partial}{\partial u_{,i}} + \sum_{1\le s}(-1)^s D_{j_1} \ldots D_{j_s} \frac{\partial}{\partial u_{,ij_1\cdots j_s}} \right\}$$
$$+ \sum_{1\le r} D_{k_1\cdots k_r}(\eta - \xi_\ell u_{,\ell}) \left\{ \frac{\partial}{\partial u_{,ik_1\cdots k_r}} + \sum_{s\ge 1}(-1)^s D_{j_1} \ldots D_{j_s} \frac{\partial}{\partial u_{,ik_1\cdots k_r j_1\cdots j_s}} \right\}. \tag{41}$$

The celebrated Noether theorem can be stated as follows.

Theorem 2.3 *(Ibragimov* [24]*). Given a formal Lie–Bäcklund transformation group with the Lie–Bäcklund operator* $\mathrm{pr}\mathbf{v}$*, the following identity*

$$\mathrm{pr}\mathbf{v} + D_i\xi_i = (\eta - \xi_j u_{,j})E + D_i N_i \tag{42}$$

holds.

Consequently, for **y** satisfying the Euler–Lagrangian equation

$$E(L([\mathbf{z}])) = 0, \tag{43}$$

the prolongation equation equals to a divergence form

$$\mathrm{pr}\mathbf{v}(L) + L\mathrm{Div}\boldsymbol{\xi} = \mathrm{Div}\mathbf{N}(L) = 0. \tag{44}$$

Taking into account null Lagrangians, there exist functions $\{B_i(\mathbf{z})\}$ such that

$$\mathrm{pr}\mathbf{v}(L) + L\mathrm{Div}\boldsymbol{\xi} = \mathrm{Div}\mathbf{B}, \tag{45}$$

then the following conservation laws yield

$$\mathrm{Div}\mathbf{P}(L) = 0, \tag{46}$$

where $P_i(L) = N_i(L) - B_i$, or $\mathbf{P}(L) = \mathbf{N}(L) - \mathbf{B}$.

3. Symmetry and Invariant Group

3.1. Lie–Bäcklund Symmetry

For planar elasticity, the Euler–Lagrangian equation of complementary energy potential is the biharmonic equation satisfied by the Airy stress function φ. Since the variational symmetry group is a subgroup of invariant transformation admitted by its Euler–Lagrangian equation, we begin by finding the Lie–Bäcklund symmetry admitted by biharmonic equations. Different from the approach adopted by Bluman and Gregory [25], we are looking for more general symmetry—the Lie–Bäcklund symmetry admitted by the biharmonic equation.

Let $\mathbf{x} = (x_1, x_2)$, $u = \varphi(x_1, x_2)$, and

$$\mathbf{x}^* = \exp(\epsilon \boldsymbol{\xi})\mathbf{x}, \tag{47}$$

$$\varphi^* = \exp(\epsilon \eta)\varphi. \tag{48}$$

The generalized vector field is given as

$$\mathbf{v} = \xi_\alpha \frac{\partial}{\partial x^\alpha} + \eta \frac{\partial}{\partial \varphi}. \tag{49}$$

Consider the infinitesimal generators of the following forms

$$\xi_\alpha = \xi_\alpha(\mathbf{x}), \tag{50}$$

$$\eta = f(\mathbf{x}) + g(\mathbf{x})\varphi + h_\gamma(\mathbf{x})\varphi_{,\gamma} + k_{\lambda\mu}(\mathbf{x})\varphi_{,\lambda\mu} + p_\lambda(\mathbf{x})\varphi_{,\alpha\alpha\lambda}, \tag{51}$$

where $f(\mathbf{x})$, $g(\mathbf{x})$ are the unknown scalar functions; $\xi_\alpha(\mathbf{x})$, $h_\alpha(\mathbf{x})$, $k_{\lambda\mu}(\mathbf{x})$, and $p_\lambda(\mathbf{x})$ are the unknown vector or tensorial functions.

By Theorem (2.2), the invariant conditions, or the determining equations, are

$$\mathrm{pr}^{(4)}\mathbf{v}\left(\frac{\partial^4 \varphi}{\partial x_\alpha \partial x_\alpha \partial x_\beta \partial x_\beta}\right) = 0, \tag{52}$$

where

$$\mathrm{pr}^{(4)}\mathbf{v} = \xi_\alpha \frac{\partial}{\partial x_\alpha} + \eta \frac{\partial}{\partial \phi} + \eta_\alpha^{(1)} \frac{\partial}{\partial \varphi_{,\alpha}} + \cdots + \eta_{\alpha\beta\lambda\mu}^{(4)} \frac{\partial}{\partial \varphi_{,\alpha\beta\lambda\mu}}. \tag{53}$$

This leads to an algebraic equation for the fourth-order extensions,

$$\eta_{\alpha\alpha\beta\beta}^{(4)} = 0. \tag{54}$$

The determining equation can be written as follows,

$$\begin{aligned}
\eta_{\alpha\beta\alpha\beta}^{(4)} = {} & f_{,\alpha\beta\alpha\beta} \\
& + [g_{,\alpha\beta\alpha\beta}\varphi + 4g_{,\alpha\beta\beta}\varphi_{,\alpha} + 2g_{,\alpha\alpha}\varphi_{,\beta\beta} + 4g_{,\alpha\beta}\varphi_{,\alpha\beta} + 4g_{,\alpha}\varphi_{,\alpha\beta\beta}] \\
& + [(h_{\gamma,\alpha\beta\alpha\beta} - \xi_{\gamma,\alpha\beta\alpha\beta})\varphi_{,\gamma} + 4(p_{\gamma,\alpha\alpha\beta} - \xi_{\gamma,\alpha\alpha\beta})\varphi_{,\gamma\beta}
\end{aligned}$$

$$+ 4(h_{\gamma,\alpha\beta} - \xi_{\gamma,\alpha\beta})\varphi_{,\gamma\alpha\beta} + 2(h_{\gamma,\alpha\alpha} - \xi_{\gamma,\alpha\alpha})\varphi_{,\gamma\beta\beta} + 4(h_{\gamma,\alpha} - \xi_{\gamma,\alpha})\varphi_{,\gamma\alpha\beta\beta}]$$
$$+ [k_{\lambda\mu,\alpha\beta\alpha\beta}\varphi_{,\lambda\mu} + 4k_{\lambda\mu,\alpha\alpha\beta}\varphi_{,\lambda\mu\beta} + 2k_{\lambda\mu,\alpha\alpha}\varphi_{,\lambda\mu\beta\beta}$$
$$+ 4k_{\lambda\mu,\alpha\beta}\varphi_{,\lambda\mu\alpha\beta} + 4k_{\lambda\mu,\beta}\varphi_{,\lambda\mu\alpha\beta\beta} + k_{\lambda\mu}\varphi_{,\lambda\mu\alpha\beta\alpha\beta}]$$
$$+ [p_{\lambda,\alpha\alpha\beta\beta}\varphi_{,\gamma\gamma\lambda} + 4p_{\lambda,\alpha\beta\beta}\varphi_{,\gamma\gamma\lambda\alpha} + 4p_{\lambda,\alpha\beta}\varphi_{,\gamma\gamma\lambda\alpha\beta}$$
$$+ 2p_{\lambda,\alpha\alpha}\varphi_{,\gamma\gamma\lambda\beta\beta} + 4p_{\lambda,\alpha}\varphi_{,\gamma\gamma\lambda\beta\beta} + p_{\lambda}\varphi_{,\gamma\gamma\lambda\alpha\alpha\beta\beta}] = 0. \tag{55}$$

Let $\bar{\xi}_\kappa := \xi_\kappa - h_\kappa$. The determining equation can be split into a set of coupled differential equations among unknown functions, $f(\mathbf{x})$, $g(\mathbf{x})$, $h_\kappa(\mathbf{x})$, $k_{\lambda\mu}(\mathbf{x})$, $p_\kappa(\mathbf{x})$, and $\xi_\kappa(\mathbf{x})$:

$$\varphi^0: \quad f_{,\alpha\alpha\beta\beta} = 0; \tag{56}$$
$$\varphi^1: \quad g_{,\alpha\alpha\beta\beta} = 0; \tag{57}$$
$$\partial\varphi: \quad 4g_{,\alpha\beta\beta}\varphi_{,\alpha} - \bar{\xi}_{\kappa,\alpha\alpha\beta\beta}\varphi_{,\kappa} = 0; \tag{58}$$
$$\partial^2\varphi: \quad 4g_{,\alpha\beta}\varphi_{,\alpha\beta} + 2g_{,\alpha\alpha}\varphi_{,\beta\beta} - \bar{\xi}_{\kappa,\alpha\beta\beta}\varphi_{,\kappa\alpha} + k_{\lambda\mu,\alpha\alpha\beta\beta}\varphi_{,\lambda\mu} = 0; \tag{59}$$
$$\partial^3\varphi: \quad 4g_{,\alpha}\varphi_{,\alpha\beta\beta} - 4\bar{\xi}_{\kappa,\alpha\beta}\varphi_{,\kappa\alpha\beta} - 2\bar{\xi}_{\kappa,\alpha\alpha}\varphi_{,\kappa\beta\beta} + 4k_{\lambda\mu,\alpha\beta\beta}\varphi_{,\lambda\mu\alpha} + p_{\kappa,\alpha\alpha\beta\beta}\varphi_{,\kappa\delta\delta} = 0; \tag{60}$$
$$\partial^4\varphi: \quad -4\bar{\xi}_{\kappa,\alpha}\varphi_{,\kappa\alpha\beta\beta} + 4k_{\lambda\mu,\alpha\beta}\varphi_{,\lambda\mu}\varphi_{,\lambda\mu\alpha\beta} + 2k_{\lambda\mu,\alpha\alpha}\varphi_{,\lambda\mu\beta\beta} + 4p_{\kappa,\alpha\beta\beta}\varphi_{,\kappa\alpha\delta\delta} = 0; \tag{61}$$
$$\partial^5\varphi: \quad 4k_{\lambda\mu,\alpha}\varphi_{,\lambda\mu\alpha\beta\beta} + 4p_{\kappa,\alpha\beta}\varphi_{,\kappa\alpha\beta\mu\mu} = 0. \tag{62}$$

A set of special solutions of the above differential equations are obtained:

$$p_\kappa(\mathbf{x}) = p^{(2)}_\kappa x_\alpha x_\alpha + p^{(1)}\epsilon_{\kappa\gamma}x_\gamma + p^{(0)}_\kappa, \tag{63}$$
$$k_{\lambda\mu}(\mathbf{x}) = k^{(2)}\delta_{\lambda\mu}x_\alpha x_\alpha + \left(k^{(1)}_\lambda x_\mu + k^{(1)}_\mu x_\lambda\right) + k^{(0)}_{\lambda\mu}, \tag{64}$$
$$\xi_\alpha(\mathbf{x}) = P^1_{\alpha\lambda\mu}x_\lambda x_\mu + \theta^1_{\alpha\beta}x_\beta + d^1_\alpha, \tag{65}$$
$$h_\alpha(\mathbf{x}) = P^2_{\alpha\lambda\mu}x_\lambda x_\mu + \theta^2_{\alpha\beta}x_\beta + d^2_\alpha. \tag{66}$$

The superscript (i) in a coefficient indicates the order of polynomial that the coefficient preceeds. The superscript $\alpha = 1, 2$, indicates different sets of coefficients.

The free parameter tensors $P^\iota_{\alpha\lambda\mu}$ and $\theta^\iota_{\alpha\beta}$ satisfy the conditions

$$P^\iota_{\alpha\lambda\mu} = P^\iota_{\alpha\mu\lambda}, \quad \iota = 1, 2 \tag{67}$$
$$P^\iota_{\alpha\lambda\mu} = -P^\iota_{\lambda\alpha\mu}, \quad \alpha \neq \lambda, \quad \iota = 1, 2 \tag{68}$$
$$\theta^\iota_{\alpha\beta}\theta^\iota_{\beta\gamma} = \delta_{\alpha\gamma}det|\theta^\iota|, \quad \text{and} \quad det|\theta^\iota| = \theta^\iota_{11}\theta^\iota_{22} - \theta^\iota_{12}\theta^\iota_{21}, \quad \iota = 1, 2. \tag{69}$$

Denote $\Delta P_{\alpha\lambda\mu} := P^1_{\alpha\lambda\mu}, -P^2_{\alpha\lambda\mu}$, $\Delta\theta_{\alpha\beta} := \theta^1_{\alpha\beta} - \theta^2_{\alpha\beta}$, $\Delta d_\alpha = d^1_\alpha - d^2_\alpha$. Then

$$\bar{\xi}_\alpha(\mathbf{x}) = \Delta P_{\alpha\lambda\mu}x_\lambda x_\mu + \Delta\theta_{\alpha\beta}x_\beta + \Delta d_\alpha,$$
$$g(\mathbf{x}) = \frac{1}{2}\bar{\xi}_{\alpha,\alpha} + c = \Delta P_{\alpha\alpha\mu}x_\mu + \frac{1}{2}\Delta\theta_{\alpha\alpha} + c, \tag{70}$$
$$f = f(\mathbf{x}), \quad \text{where} \quad f_{,\alpha\alpha\beta\beta} = 0, \tag{71}$$

and $f(\mathbf{x})$ can be any function that satisfies the biharmonic equation.

3.2. Variational Symmetry

The symmetry group admitted by the Euler–Lagrangian equation may not necessarily yield variational symmetry. A simple procedure to find the variational symmetry group is to test all invariant solutions admitted by the Euler–Lagrangian equation and to select those that indeed satisfy both the determining equation and the prolongation equation, i.e.

$$\begin{aligned}
&\mathrm{pr}\mathbf{v}L_c + (D_\alpha \xi_\alpha)L_c = \eta^{(2)}_{\alpha\beta}\frac{\partial L_c}{\partial \varphi_{,\alpha\beta}} + (D_\alpha \xi_\alpha)L_c = 0\\
&\Rightarrow (D_\alpha D_\beta(\eta - \xi_{,\gamma}\varphi_{,\gamma}) + \xi_\gamma \varphi_{,\gamma\alpha\beta})\frac{\partial L_c}{\partial \varphi_{,\alpha\beta}} + \xi_{\gamma,\gamma}L_c\\
&= \frac{\partial L_c}{\partial \varphi_{,\alpha\beta}}\{f_{,\alpha\beta} + (g_{,\alpha\beta}\varphi + g_{,\alpha}\varphi_{,\beta} + g_{,\beta}\varphi_{,\alpha} + g\varphi_{,\alpha\beta})\\
&\quad + h_\kappa \varphi_{,\kappa\alpha\beta} - (\bar{\xi}_{\kappa,\alpha\beta}\varphi_{,\kappa} + \bar{\xi}_{\kappa,\alpha}\varphi_{,\kappa\beta} + \bar{\xi}_{\kappa,\beta}\varphi_{,\kappa\alpha})\\
&\quad + (+k_{\lambda\mu,\alpha}\varphi_{,\lambda\mu\beta} + k_{\lambda\mu,\beta}\varphi_{,\lambda\mu\alpha} + k_{\lambda\mu}\varphi_{,\lambda\mu\alpha\beta})\\
&\quad + (p_{\kappa,\alpha\beta}\varphi_{,\kappa\gamma\gamma} + p_{\kappa,\alpha}\varphi_{,\kappa\beta\gamma\gamma} + p_{\kappa,\beta}\varphi_{,\kappa\alpha\gamma\gamma} + p_\kappa \varphi_{,\kappa\alpha\beta\gamma\gamma})\} &(72)\\
&\quad + \frac{1}{2E}((1+\nu)\varphi_{,\alpha\alpha}\varphi_{,\alpha\alpha} - \nu\varphi_{,\alpha\alpha}\varphi_{,\beta\beta})\xi_{\gamma,\gamma} = 0, &(73)
\end{aligned}$$

where

$$\frac{\partial L_c}{\partial \varphi_{,\alpha\beta}} = \left(\frac{1+\nu}{E}\varphi_{,\alpha\beta} - \frac{\nu}{E}\varphi_{,\tau\tau}\delta_{\alpha\beta}\right). \tag{74}$$

Consequently, one obtains the following Killing's equations,

$$\begin{aligned}
&\partial^2\varphi: &&\frac{\partial L_c}{\partial \varphi_{,\alpha\beta}} f_{,\alpha\beta} = 0; &(75)\\
&\varphi\partial^2\varphi: &&\frac{\partial L_c}{\partial \varphi_{,\alpha\beta}}(\varphi g_{,\alpha\beta}) = 0; &(76)\\
&\partial\varphi\partial^2\varphi: &&\frac{\partial L_c}{\partial \varphi_{,\alpha\beta}}(g_{,\alpha}\varphi_{,\beta} + g_{,\beta}\varphi_{,\alpha} - \bar{\xi}_{\kappa,\alpha\beta}\varphi_\kappa) = 0; &(77)\\
&\partial^2\varphi\partial^2\varphi: &&\frac{\partial L_c}{\partial \varphi_{,\alpha\beta}}\left(g\varphi_{,\alpha\beta} - \bar{\xi}_{\kappa,\alpha}\varphi_{,\kappa\beta} - \bar{\xi}_{\kappa,\beta}\varphi_{,\kappa\alpha} + \frac{1}{2}\xi_{\gamma,\gamma}\varphi_{,\alpha\beta}\right) = 0;\\
&\partial^2\varphi\partial^3\varphi: &&\frac{\partial L_c}{\partial \varphi_{,\alpha\beta}}(h_\kappa\varphi_{,\kappa\alpha\beta} + k_{\lambda\mu,\alpha}\varphi_{,\lambda\mu\beta} + k_{\lambda\mu,\beta}\varphi_{,\lambda\mu\alpha} + p_{\kappa,\alpha\beta}\varphi_{,\kappa\gamma\gamma}) = 0; &(78)\\
&\partial^2\varphi\partial^4\varphi: &&\frac{\partial L_c}{\partial \varphi_{,\alpha\beta}}(k_{\lambda\mu}\varphi_{,\lambda\mu\alpha\beta} + p_{\lambda,\alpha}\varphi_{,\lambda\beta\gamma\gamma} + p_{\lambda,\beta}\varphi_{,\lambda\alpha\gamma\gamma}) = 0; &(79)\\
&\partial^2\varphi\partial^5\varphi: &&\frac{\partial L_c}{\partial \varphi_{,\alpha\beta}} p_\kappa\varphi_{,\kappa\alpha\beta\mu\mu} = 0. &(80)
\end{aligned}$$

By substituting solutions (63)–(66) and (66)–(71) in the above equations, one may find the following additional constraints:

$$\frac{\partial^2 f}{\partial x_\alpha \partial x_\beta} = 0, \tag{81}$$

$$h_\alpha = 0, \tag{82}$$

$$k_{\alpha\beta} = 0, \tag{83}$$

$$p_\alpha = 0. \tag{84}$$

Thus, the variational invariant transformations are[1]

$$\xi_1 = a_1 x_1 - a_2 x_2 + a_3, \tag{85}$$

$$\xi_2 = a_2 x_1 + a_1 x_2 + a_4, \tag{86}$$

$$\eta = (a_5 x_1 + a_6 x_2 + a_7) + a_1 \varphi, \tag{87}$$

where a_i are arbitrary constants. Note that Equations (85) and (86) may be written as $\xi_\alpha = a_1 x_\alpha + a_2 \epsilon_{\alpha\beta} x_\beta + b_\alpha$, with $b_1 = a_3$ and $b_2 = a_4$.

When $\nu = 1/2$ for plane strain case, we have additional inversion transformations

$$\xi_1 = a_8 \left(x_1^2 - x_2^2\right) + 2a_9 x_1 x_2, \tag{88}$$

$$\xi_2 = 2a_8 x_1 x_2 - a_9 \left(x_1^2 - x_2^2\right), \tag{89}$$

$$\eta = (2a_8 x_1 - 2a_9 x_2)\varphi. \tag{90}$$

Note that a similar solution can be obtained when $\nu = 1$ for plane stress. Since Poisson's ratio cannot be greater than $1/2$, an invariant solution at $\nu = 1$ is not realistic.

Therefore, the Lie group of variational invariant transformation, i.e., the tangent vector fields, is

$$X_1 = -x_2 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2}, \qquad X_2 = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + \varphi \frac{\partial}{\partial \varphi},$$
$$X_3 = \frac{\partial}{\partial x_1}, \qquad X_4 = \frac{\partial}{\partial x_2}, \qquad X_5 = x_1 \frac{\partial}{\partial \varphi}, \qquad X_6 = x_2 \frac{\partial}{\partial \varphi}, \qquad X_7 = \frac{\partial}{\partial \varphi}. \tag{91}$$

When $\nu = 1/2$, there are two additional invariant vector fields for plane strain state

$$X_8 = \left(x_1^2 - x_2^2\right) \frac{\partial}{\partial x_1} + 2x_1 x_2 \frac{\partial}{\partial x_2} + 2x_1 \varphi \frac{\partial}{\partial \varphi},$$
$$X_9 = -2x_1 x_2 \frac{\partial}{\partial x_1} + \left(x_1^2 - x_2^2\right) \frac{\partial}{\partial x_2} - 2x_2 \varphi \frac{\partial}{\partial \varphi}.$$

3.3. Divergence Symmetry

The variational symmetry group found is only a subgroup of point transformation, as shown in (91). However, the generalized transformations, or Lie–Bäcklund transformations, can have divergence symmetry. There exist functions, B_α, such that the Noether theorem holds in the following Bessel–Hagen form,

$$\mathrm{pr}\mathbf{v}(L_c) + (D_\alpha \xi_\alpha) L_c = D_\alpha B_\alpha. \tag{92}$$

In the following, several divergence symmetric transformations are found.

[1] There may exist some higher-order variational symmetry.

i) Divergence symmetry I

Consider the following proper Lie–Bäcklund transformation, which is admitted by the biharmonic equation,

$$\xi_\alpha = 0, \qquad \text{and} \qquad \eta = \psi(x_1, x_2), \tag{93}$$

where the function ψ is an arbitrary solution of biharmonic equation. Choose

$$B_\alpha = h_\beta \vartheta_{,\gamma\gamma\alpha}\varphi_{,\beta} \tag{94}$$

where h_β is a constant vector and ϑ is another solution of the biharmonic equation, i.e. $\vartheta_{,\alpha\alpha\beta\beta} = 0$. Consequently,

$$D_\alpha B_\alpha = h_\beta \vartheta_{,\gamma\gamma\alpha}\varphi_{,\alpha\beta}. \tag{95}$$

Hence, by Noether identity (42), one may find that

$$\psi_{,\alpha\beta} = E_{\alpha\beta\lambda\mu} h_\mu \vartheta_{,\gamma\gamma\lambda}, \tag{96}$$

or vice versa,

$$h_\beta \vartheta_{,\gamma\gamma\alpha} = C_{\alpha\beta\lambda\mu}\psi_{,\lambda\mu}, \tag{97}$$

$$B_\alpha = C_{\alpha\beta\lambda\mu}\psi_{,\lambda\mu}\varphi_{,\beta}, \tag{98}$$

where $E_{\alpha\beta\lambda\mu}$ and $C_{\alpha\beta\lambda\mu}$ are the elastic stiffness tensor and elastic compliance tensor defined in Equations (2) and (3).

It is worth verifying that indeed,

$$\psi_{,\alpha\alpha\beta\beta} = h_\mu \left(\frac{E}{1+\nu} + \frac{2E\nu}{1-\nu^2} \right) \vartheta_{,\gamma\gamma\mu\beta\beta} = 0. \tag{99}$$

A tangent vector field with divergence symmetry is found to be

$$X_I = \psi(x_1, x_2)\frac{\partial}{\partial \varphi}, \tag{100}$$

with $\psi_{,\alpha\alpha\beta\beta} = 0$.

ii) Divergence symmetry II

Let

$$\xi_\alpha = 0, \qquad \text{and} \qquad \eta = b_{II\gamma}\varphi_{,\gamma}, \tag{101}$$

where $\{b_{II\gamma}\}$ is a constant vector. It can be readily shown that

$$\mathbf{prv}(L_c) + (D_\alpha \xi_\alpha) L_c = \eta^{(2)}_{\alpha\beta}\frac{\partial L_c}{\partial \varphi_{,\alpha\beta}} = b_{II\gamma}\frac{\partial}{\partial x_\gamma}\left\{ C_{\alpha\beta\lambda\mu}\varphi_{,\alpha\beta}\varphi_{,\lambda\mu} \right\}. \tag{102}$$

Choose

$$B_\alpha = \frac{1}{2} b_{II\alpha} C_{\lambda\mu\gamma\delta} \varphi_{\lambda\mu} \varphi_{\gamma\delta}. \tag{103}$$

The divergence symmetric tangent vector field is

$$X_{II1} = \varphi_{,1} \frac{\partial}{\partial \varphi}, \tag{104}$$

$$X_{II2} = \varphi_{,2} \frac{\partial}{\partial \varphi}. \tag{105}$$

This divergence symmetry is equivalent to the variational symmetry due to coordinate translation.
iii) Divergence symmetry III
Let

$$\xi_\alpha = 0 \qquad \text{and} \qquad \eta = b_{III}(\varphi - x_\gamma \varphi_{,\gamma}), \tag{106}$$

where b_{III} is an arbitrary constant. It is straightforward to verify that

$$D_\alpha D_\beta \eta = -b_{III}(\varphi_{,\alpha\beta} + x_\gamma \varphi_{,\gamma\alpha\beta}) \tag{107}$$

$$\mathrm{pr}\mathbf{v}(L_c) + (D_\alpha \xi_\alpha) L_c = -\frac{b_{III}}{2} \frac{\partial}{\partial x_\gamma} \{C_{\alpha\beta\lambda\mu} \varphi_{,\alpha\beta} \varphi_{,\lambda\mu} x_\gamma\}. \tag{108}$$

Choose $B_\alpha = -a_{II} L_c x_\gamma$. The following vector field is divergence symmetric,

$$X_{III} = (\varphi - x_\gamma \varphi_{,\gamma}) \frac{\partial}{\partial \varphi}. \tag{109}$$

This divergence symmetry is equivalent to the variational symmetry due to scaling.
iv) Divergence symmetry IV
Let

$$\xi_\alpha = 0, \qquad \text{and} \qquad \eta = b_{IV} \epsilon_{\lambda\mu} x_\lambda \varphi_{,\mu}. \tag{110}$$

One may find that

$$\begin{aligned} \eta^{(2)}_{\alpha\beta} \frac{\partial L_c}{\partial \varphi_{,\alpha\beta}} &= b_{IV} \left(\frac{1+\nu}{E} \varphi_{,\alpha\beta} - \frac{\nu}{E} \varphi_{,\gamma\gamma} \delta_{\alpha\beta} \right) (\epsilon_{\lambda\mu} x_\lambda \varphi_{,\mu\alpha\beta}) \\ &= \frac{b_{IV}}{2} \frac{d}{dx_\mu} \left\{ \frac{1+\nu}{E} (\epsilon_{\lambda\mu} x_\lambda \varphi_{,\alpha\beta} \varphi_{,\alpha\beta}) - \frac{\nu}{E} \left(\epsilon_{\lambda\mu} x_\lambda \varphi^2_{,\gamma\gamma} \right) \right\}. \end{aligned} \tag{111}$$

Choose

$$B_\alpha = -\frac{b_{IV}}{2} \left\{ \frac{1+\nu}{E} \left(\epsilon_{\alpha\beta} x_\beta \varphi_{,\lambda\mu} \varphi_{,\lambda\mu} \right) + \frac{\nu}{E} \left(\epsilon_{\alpha\beta} x_\beta \varphi^2_{,\gamma\gamma} \right) \right\}. \tag{112}$$

The divergence symmetric tangent vector field is

$$X_{IV} = \epsilon_{\alpha\beta} x_\beta \varphi_{,\alpha} \frac{\partial}{\partial \varphi}. \tag{113}$$

Again, this vector field belongs to the equivalent class of a variational symmetry due to coordinate rotation.

v) Divergence symmetry V

Let

$$\xi_\alpha = 0, \quad \text{and} \quad \eta = b_V \varphi_{,\lambda\lambda}. \tag{114}$$

Subsequently

$$\eta^{(2)}_{\alpha\beta} = b_V \varphi_{,\lambda\lambda\alpha\beta}, \tag{115}$$

and

$$\eta^{(2)}_{\alpha\beta} \frac{\partial L_c}{\varphi_{,\alpha\beta}} = b_V \frac{1+\nu}{E} \frac{\partial}{\partial x_\alpha} (\varphi_{,\beta} \varphi_{,\lambda\lambda\alpha\beta}). \tag{116}$$

Choose

$$B_\alpha = b_V \frac{1+\nu}{E} \varphi_{,\lambda\lambda\alpha\beta} \varphi_{,\beta}. \tag{117}$$

The divergence symmetric tangent vector field is

$$X_V = \varphi_{,\lambda\lambda} \frac{\partial}{\partial \varphi}. \tag{118}$$

vi) Divergence symmetry VI

Let

$$\xi_\alpha = 0, \quad \text{and} \quad \eta = b_{VI\lambda} \varphi_{,\lambda\mu\mu}, \tag{119}$$

where $\{b_{VI\lambda}\}$ is a constant vector. Hence

$$\eta^{(2)}_{\alpha\beta} = b_{VI\lambda} \varphi_{,\lambda\mu\mu\alpha\beta}. \tag{120}$$

Consequently,

$$\eta^{(2)}_{\alpha\beta} \frac{\partial L_c}{\varphi_{,\alpha\beta}} = b_{VI\lambda} \frac{1+\nu}{E} \frac{\partial}{\partial x_\alpha} (\varphi_{,\beta} \varphi_{,\lambda\mu\mu\alpha\beta}). \tag{121}$$

Choose

$$B_\alpha = b_{VI\lambda} \frac{1+\nu}{E} \varphi_{,\beta} \varphi_{,\lambda\mu\mu\alpha\beta}. \tag{122}$$

The corresponding divergence invariant vector field is

$$X_{VI1} = \varphi_{,1\mu\mu}\frac{\partial}{\partial\varphi}, \tag{123}$$

$$X_{VI2} = \varphi_{,2\mu\mu}\frac{\partial}{\partial\varphi}. \tag{124}$$

vii) Divergence symmetry VII

Let

$$\xi_\alpha = 0 \qquad \text{and} \qquad \eta = b_{VII}\epsilon_{\kappa\gamma}x_\gamma\varphi_{,\kappa\mu\mu}. \tag{125}$$

It can be readily shown that

$$\eta^{(2)}_{\alpha\beta}\frac{\partial L_c}{\varphi_{,\alpha\beta}} = b_{VII}\frac{1+\nu}{E}\frac{\partial}{\partial x_\alpha}\{2[\epsilon_{\kappa\beta}\varphi_{,\mu\mu\kappa\alpha}\varphi_{,\beta}] + [\epsilon_{\kappa\gamma}x_\gamma\varphi_{,\lambda\mu\mu\alpha\beta}\varphi_{,\beta}]\}. \tag{126}$$

Choose

$$B_\alpha = b_{VII}\frac{1+\nu}{E}\{2[\epsilon_{\kappa\beta}\varphi_{,\mu\mu\kappa\alpha}\varphi_{,\beta}] + [\epsilon_{\kappa\gamma}x_\gamma\varphi_{,\lambda\mu\mu\alpha\beta}\varphi_{,\beta}]\}. \tag{127}$$

The divergence invariant vector field is

$$X_{VII} = \epsilon_{\kappa\gamma}x_\gamma\varphi_{,\kappa\mu\mu}\frac{\partial}{\partial\varphi}. \tag{128}$$

Remark 3.1.

- The above calculation is demonstrative; we have not exhausted, in any way, the possibilities of divergence symmetry.
- There could be a confusion regarding the relationship between null Lagrangian and natural boundary conditions. In our problem, prescribed stress function on the boundary will mean Dirichlet boundary condition, even though it is stress type of boundary condition in physics. In other words, there is a difference between the Neumann boundary condition in mathematics and the ‘natural boundary condition’ in physics.

4. Dual Conservation Laws

There are two groups of dual conservation laws:

- The genuine variational-symmetric conservation laws:

$$P^{(var)}_\alpha = L_c\xi_\alpha - \frac{1}{E}(\eta - \xi_\gamma\varphi_{,\gamma})\varphi_{,\beta\beta\alpha} + D_\beta(\eta - \xi_\gamma\varphi_{,\gamma})\frac{\partial L_c}{\partial\varphi_{,\alpha\beta}}; \tag{129}$$

- The generalized (divergence-symmetric) conservation laws:

$$P^{(div)}_\alpha = L_c\xi_\alpha - \frac{1}{E}(\eta - \xi_\gamma\varphi_{,\gamma})\varphi_{,\beta\beta\alpha} + D_\beta(\eta - \xi_\gamma\varphi_{,\gamma})\frac{\partial L_c}{\partial\varphi_{,\alpha\beta}} - B_\alpha. \tag{130}$$

4.1. Variational-Invariant Conservation Laws

Variational dual conservation laws in planar elasticity are listed as follows.

1) Scaling.

Let $a_1 = 1$, and $a_i = 0,\ i \neq 1$ in variational symmetric transformations (85)–(90). One then has

$$\xi_\alpha = x_\alpha, \qquad \eta = \varphi; \tag{131}$$

$$\xi_{\alpha,\beta} = \delta_{\alpha\beta}, \qquad \eta_\beta = \varphi_{,\beta}. \tag{132}$$

Denote

$$\mathcal{S}_\alpha := P_\alpha^{(var)}\Big|_{\{a_1 \neq 0\}}. \tag{133}$$

One may find that

$$\mathcal{S}_\alpha = L_c x_\alpha - \frac{1}{E}(\varphi - x_\lambda \varphi_{,\lambda})\varphi_{,\beta\beta\alpha} - x_\gamma \varphi_{,\gamma\beta}\left(\frac{1+\nu}{E}\varphi_{,\alpha\beta} - \frac{\nu}{E}\varphi_{,\tau\tau}\delta_{\alpha\beta}\right). \tag{134}$$

2) Coordinate rotation.

Let $a_2 = 1$ and $a_i = 0\ \ (i \neq 2)$ in Equations (85)–(90). One has

$$\xi_\alpha = \epsilon_{\alpha\beta} x_\beta, \qquad \eta = 0; \tag{135}$$

$$\eta - \xi_\gamma \varphi_{,\gamma} = -\epsilon_{\gamma\lambda} x_\lambda \varphi_{,\gamma}, \qquad D_\beta(\eta - \xi_\gamma \varphi_{,\gamma}) = -\epsilon_{\gamma\beta}\varphi_{,\gamma} - \epsilon_{\gamma\lambda} x_\lambda \varphi_{,\gamma\beta}. \tag{136}$$

Denote

$$\mathcal{R}_\alpha := P_\alpha^{(var)}\Big|_{\{a_2 \neq 0\}}. \tag{137}$$

One may find that

$$\begin{aligned}\mathcal{R}_\alpha = {} & L_c \epsilon_{\alpha\beta} x_\beta + \frac{1}{E}\epsilon_{\gamma\lambda} x_\lambda \varphi_{,\gamma}\varphi_{,\beta\beta\alpha} \\ & - (\epsilon_{\gamma\beta}\varphi_{,\gamma} + \epsilon_{\gamma\lambda} x_\lambda \varphi_{,\gamma\beta})\left(\frac{1+\nu}{E}\varphi_{,\alpha\beta} - \frac{\nu}{E}\varphi_{,\tau\tau}\delta_{\alpha\beta}\right).\end{aligned} \tag{138}$$

3) Coordinate translation.

Let $a_3 = \delta_{1\kappa},\ a_4 = \delta_{2\kappa}$, and $a_i = 0,\ i \neq 3, 4$ in transformations (85)–(90). Consequently,

$$\xi_\alpha = \delta_{\alpha\kappa}, \qquad \eta = 0. \tag{139}$$

Define

$$\mathcal{T}_{\alpha(\kappa)} := P_\alpha^{(var)}\Big|_{\{a_3, a_4 \neq 0\}}. \tag{140}$$

It follows that

$$\mathcal{T}_{\alpha(\kappa)} = L_c \delta_{\alpha\kappa} + \frac{1}{E}\varphi_{,\kappa}\varphi_{,\beta\beta\alpha} - \varphi_{,\kappa\beta}\left(\frac{1+\nu}{E}\varphi_{,\alpha\beta} - \frac{\nu}{E}\varphi_{,\tau\tau}\delta_{\alpha\beta}\right) \tag{141}$$

where $\alpha, \kappa = 1, 2$.

Remark 4.1. The divergence-free second-order tensor $\mathcal{T}_{\alpha(\kappa)}$ may be called the dual-Eshelby tensor. The following integral

$$J_\kappa^* = \int_{\partial\Omega}\left(L_c n_\kappa + \frac{1}{E}\varphi_{,\kappa}\varphi_{,\beta\beta\alpha}n_\alpha - \varphi_{,\kappa\beta}\left(\frac{1+\nu}{E}\varphi_{,\alpha\beta} - \frac{\nu}{E}\varphi_{,\tau\tau}\delta_{\alpha\beta}\right)n_\alpha\right)dS, \tag{142}$$

may be called the dual J-integral.

4) Compatibility identities.

Let $a_5 = a_6 = 1$ and $a_i = 0, \; i \neq 5, 6$. We may assume that

$$\xi_\alpha = 0, \qquad \eta = \delta_{\alpha\kappa}x_\alpha, \tag{143}$$

where κ is a fixed number.

Let

$$\mathcal{C}_{\alpha(\kappa)} := P_\alpha^{(var)}\Big|_{\{a_5 = a_6 = 1, \text{ and } a_i = 0, \; i \neq 5,6\}}. \tag{144}$$

We have the dual conservation law,

$$\mathcal{C}_{\alpha(\kappa)} = \frac{1}{E}x_\kappa\varphi_{,\beta\beta\alpha} + \delta_{\beta\kappa}\left(\frac{1+\nu}{E}\varphi_{,\alpha\beta} - \frac{\nu}{E}\varphi_{,\tau\tau}\delta_{\alpha\beta}\right), \tag{145}$$

where $\alpha, \kappa = 1, 2$.

5) Gauss theorem (divergence theorem).
Assume $a_7 = 1$ and $a_i = 0, \; i \neq 7$. Then

$$\xi_\alpha = 0; \qquad \eta = 1. \tag{146}$$

Let

$$\mathcal{G}_\alpha := P_\alpha^{(var)}\Big|_{\{a_7 = 1\}}. \tag{147}$$

We recover the Gauss (divergence) theorem

$$\mathcal{G}_\alpha = \frac{1}{E}\varphi_{,\beta\beta\alpha}. \tag{148}$$

6) Inversion.

When $\nu = 1/2$ for plane strain state, additional conservation laws is valid. Assume $a_8 = \delta_{1\kappa}, a_9 = \delta_{2\kappa}$ for a fixed κ, and $a_i = 0,\ i \neq 8, 9$. The infinitesimal generators have the forms

$$\xi_\alpha = 2x_\alpha x_\kappa - \delta_{\alpha\kappa} x_\lambda x_\lambda, \tag{149}$$

$$\eta = 2x_\kappa \varphi. \tag{150}$$

Subsequently,

$$\eta - \xi_\gamma \varphi_{,\gamma} = 2x_\kappa \varphi - (2x_\gamma x_\kappa - \delta_{\gamma\kappa} x_\lambda x_\lambda)\varphi_{,\gamma}, \tag{151}$$

$$\begin{aligned} D_\beta(\eta - \xi_\gamma \varphi_{,\gamma}) = {} & 2\delta_{\beta\kappa}\varphi - 2(\delta_{\beta\kappa} x_\gamma - \delta_{\gamma\kappa} x_\beta)\varphi_{,\gamma} \\ & -2(2x_\gamma x_\kappa - \delta_{\gamma\kappa} x_\lambda x_\lambda)\varphi_{,\gamma\beta}. \end{aligned} \tag{152}$$

Define

$$\mathcal{I}_{\alpha(\kappa)} := P_\alpha^{(var)}\Bigg|_{\{a_8 \neq 0, a_9 \neq 0\}}. \tag{153}$$

It then follows that

$$\begin{aligned} \mathcal{I}_{\alpha(\kappa)} = {} & (2x_\alpha x_\kappa - \delta_{\alpha\kappa} x_\lambda x_\lambda) L_c \\ & - \frac{(1-\nu^2)}{E}(2x_\kappa \varphi - (2x_\kappa x_\gamma - \delta_{\gamma\kappa} x_\lambda x_\lambda))\varphi_{,\gamma}\varphi_{,\beta\beta\alpha} \\ & + \frac{(1+\nu)}{E}\{2\delta_{\beta\kappa}\varphi - 2(\delta_{\beta\kappa} x_\gamma + \delta_{\gamma\kappa} x_\beta)\varphi_{,\gamma} \\ & - (2x_\gamma x_\kappa - \delta_{\gamma\kappa} x_\lambda x_\lambda)\varphi_{,\beta\gamma}\}(\varphi_{,\alpha\beta} - \nu\varphi_{,\tau\tau}\delta_{\alpha\beta}), \end{aligned} \tag{154}$$

where $\alpha, \kappa = 1, 2$.

4.2. Bessel–Hagen Type Conservation Laws

The dual conservation laws in this category involve a term, B_α, which can be introduced by a null Lagrangian. A few examples of Bessel–Hagen type conservation laws are presented in the following.

1) Reciprocal formula of Betti–Rayleigh type.

Let

$$\xi_\alpha = 0, \quad \eta = \psi, \qquad \text{and} \qquad B_\alpha = C_{\alpha\beta\lambda\mu}\psi_{,\lambda\mu}\varphi_{,\beta}, \tag{155}$$

where ψ is a solution of biharmonic equation, i.e. $\psi_{,\mu\mu\lambda} = 0$. We then have the following Betti–Rayleigh type reciprocal formula,

$$\mathcal{H}_{\alpha}^{(I)} = -\frac{1}{E}\psi\varphi_{,\beta\beta\alpha} + \psi_{,\beta}\left(\frac{1+\nu}{E}\varphi_{,\alpha\beta} - \frac{\nu}{E}\varphi_{,\tau\tau}\delta_{\alpha\beta}\right) - \varphi_{,\beta}\left(\frac{1+\nu}{E}\psi_{,\alpha\beta} - \frac{\nu}{E}\psi_{,\tau\tau}\delta_{\alpha\beta}\right). \tag{156}$$

2) Higher-order conservation law (II)

Let

$$\xi_{\alpha} = 0, \qquad \eta = \varphi_{,\gamma\gamma}. \tag{157}$$

Hence $\eta_{\beta} = \varphi_{,\gamma\gamma\beta}$. Choose

$$B_{\alpha} = C_{\alpha\beta\lambda\mu}\varphi_{,\beta}\varphi_{,\gamma\gamma\lambda\mu}. \tag{158}$$

It then follows

$$\mathcal{H}_{\alpha}^{(II)} = -\frac{1}{E}\varphi_{,\gamma\gamma}\varphi_{,\beta\beta\alpha} + C_{\alpha\beta\lambda\mu}\varphi_{,\lambda\mu}\varphi_{,\gamma\gamma\beta} - C_{\alpha\beta\lambda\mu}\varphi_{,\gamma\gamma\lambda\mu}\varphi_{,\beta}. \tag{159}$$

3) Higher-order conservation law (III).

Let

$$\xi_{\alpha} = 0, \qquad \eta = \delta_{\lambda\kappa}\varphi_{,\lambda\mu\mu} = \varphi_{,\kappa\mu\mu}, \tag{160}$$

for a fixed number κ: 1 or 2. Therefore $\eta_{,\beta} = \varphi_{,\mu\mu\beta\kappa}$. The corresponding null divergence is

$$B_{\alpha(\kappa)} = C_{\alpha\beta\lambda\mu}\varphi_{,\lambda\mu\gamma\gamma\kappa}\varphi_{,\beta}. \tag{161}$$

We have the following conservation law

$$\mathcal{H}_{\alpha(\kappa)}^{(III)} = -\frac{1}{E}\varphi_{,\lambda\lambda\kappa}\varphi_{,\beta\beta\alpha} + C_{\alpha\beta\lambda\mu}\varphi_{,\lambda\mu}\varphi_{,\mu\mu\beta\kappa} - C_{\alpha\beta\lambda\mu}\varphi_{,\lambda\mu\gamma\gamma\kappa}\varphi_{,\beta}. \tag{162}$$

4) Higher-order conservation law (IV).

Let

$$\xi_{\alpha} = 0, \quad \text{and} \quad \eta = \epsilon_{\kappa\gamma}x_{\gamma}\varphi_{,\kappa\mu\mu}. \tag{163}$$

Choosing

$$B_{\alpha} = \frac{1+\nu}{E}(2[\epsilon_{\kappa\beta}\varphi_{,\mu\mu\kappa\alpha}\varphi_{,\beta}] + [\epsilon_{\kappa\gamma}x_{\gamma}\varphi_{,\lambda\mu\mu\alpha\beta}\varphi_{,\beta}]), \tag{164}$$

we have the higher-order conservation law

$$\mathcal{H}_\alpha^{(IV)} = -\frac{1}{E}\epsilon_{\kappa\gamma}x_\gamma\varphi_{,\kappa\mu\mu}\varphi_{,\beta\beta\alpha} + C_{\alpha\beta\lambda\mu}\varphi_{,\lambda\mu}(\epsilon_{\kappa\beta}\varphi_{,\kappa\mu\mu} + \epsilon_{\kappa\gamma}x_\gamma\varphi_{,\lambda\lambda\kappa\beta})$$
$$- \frac{1+\nu}{E}(2[\epsilon_{\kappa\beta}\varphi_{,\mu\mu\kappa\alpha}\varphi_{,\beta}] + [\epsilon_{\kappa\gamma}x_\gamma\varphi_{,\kappa\mu\mu\alpha\beta}\varphi_{,\beta}]). \quad (165)$$

5. Closure

Most dual conservation laws derived in this paper are new, except the conservation laws corresponding to Gauss (divergence) theorem and Betti–Raylaigh reciprocal formula.

Using stress function formalism to derive conservation laws has practical interests. It is well known that the Airy stress function can be expressed by two analytical functions, a fact guaranteed by Goursat's theorem (see Muskhelishvili [26]). In fact, the stress function related complex variable formulation has been a primary method used to solve many engineering problems such as crack problems, and the success of fracture mechanics owes a great deal to it. The dual conservation laws obtained in this paper may allow us to make an easy link between invariant path-integrals and Muskhelishvili's complex potentials.

The same procedure can be readily extended into three-dimensional (3D) elasticity. Dual conservation laws can be derived based on general Maxwell–Morera stress function formalism. The 3D extension of this paper will be discussed in a separate paper.

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