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On dual conservation laws in planar elasticity

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Abstract

Dual conservation laws of linear planar elasticity theory have been systematically studied based on stress function formalism. By employing generalized symmetry transformation or Lie–Bäcklund transformation, a class of new dual conservation laws in planar elasticity have been discovered based on Noether theorem and its Bessel–Hagen generalization. These dual conservation laws represent variational symmetry properties of complementary potential energy, which stems from the symmetry properties of compatibility conditions—a biharmonic equation in two dimension. The physical implications of these dual conservation laws are discussed briefly. In particular, a dual-Eshelby tensor is constructed and compared with the Eshelby's energy momentum tensor.

Keywords: Conservation laws; Elasticity; Lie–Bäcklund transform; Path-independent integrals

1. Introduction

The conservation law of elasticity has been well studied in past thirty years, though the origin of its intellectual inspiration may be traced back to Eshelby's seminal work in 1950s [7,8]. Since Rice [29] linked J-integral with the energy release rate of a crack, the subject has then become part of the theoretical foundation of fracture mechanics. Landmark contributions on conservation laws of elasticity include: Günther [14], Knowles and Sternberg [15], Budiansky and Rice [2], Eshelby [10], Fletcher [12], Edelen [6], Olver [25,26,28], Suhubi [30] and among others.

Classical elasticity is a perfect embodiment of duality, in which strain representation and stress representation complement each other to describe a complete image of equilibrium-deformation

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process. On variational level, the duet are: the minimal potential energy principle and the minimal complementary energy principle. Since conservation laws of elasticity are manifestation of symmetry properties of variational principles in elasticity, naturally, conservation laws of elasticity ought to come as dual pairs, and they should be displayed with equal footing. Indeed, some authors have studied conservation laws based on complementary variational principle. Several dual invariant integrals or dual conservation laws have been derived. Among them, the dual J integral derived by Bui [3] is the earliest contribution. Other notable contributions include Sun [31] and Li [21].

The early studies on dual conservation laws are mainly based on physical observation or intuition via direct divergence-free inspection. The path integrals derived are invariant indeed. However, most early studies are not only incomplete, but also do not match the standard of elegance and rigor that are usually expected in continuum mechanics. Part of the reason may be attributed to lack of serious attention on the subject. Probably the lack of proper physical interpretation of dual conservation laws is another reason attributed for such public oblivion. In fact, most of dual conservation laws published in literature are trivial in the sense that they can be easily obtained by integration by parts from the conservation laws of the potential energy variational principle, which may give an impression that the conservation laws of the Navier equations may have exhausted all the possible symmetry properties of linear elasticity. The dual conservation laws may be just repetition of the conservation law derived from the minimal potential energy principle, and no more non-trivial conservation laws undiscovered in linear elasticity theory.

Apparently, this is a false impression. Over the years, additional conservation laws of two-dimensional elasticity have been discovered, e.g. conservation laws derived by Christiansen et al. [4,5], Falvin [11], Horgan et al. [16,17], and Miller and Horgan [22]. In fact, these conservation laws are very useful as theoretical apparatuses in estimating energy bounds, justifying the Saint-Venant principle, and possibly in convergence study of finite element methods.

As well understood, the variational symmetry group of the minimal potential energy principle is a subgroup of the symmetry transformation group of the Navier equations (See [25,26]), which characterize the symmetry properties of equilibrium equations, or equation of motion. Following the same logic, the variational symmetry group of minimal complementary energy principle should be the subgroup of symmetry transformation group of compatibility equations, which characterize the intrinsic symmetry properties of deformation. Eshelby [9] remarked: *The natural arguments of the complementary energy are the stress. To fit the formalism I presented, the stress would have to be written as the gradient of something. I dare say that if this were done in detail something interesting might come out...* Following Eshelby's suggestion, dual conservation laws in planar elasticity are studied in this paper based on stress function formalism.

2. Preliminary

2.1. Planar elasticity

To fix the notation, we start with reviewing some basic facts of planar elasticity. For plane stress state, linear two-dimensional (2D) elastic constitutive relations can be written,

$$\varepsilon_{\alpha\beta} = C_{\alpha\beta\lambda\mu}\sigma_{\lambda\mu} \quad \text{or} \quad \sigma_{\alpha\beta} = E_{\alpha\beta\lambda\mu}\varepsilon_{\lambda\mu}, \tag{2.1}$$

where $\varepsilon_{\alpha\beta}$, $\sigma_{\alpha\beta}$ are the usual strain, stress tensor respectively; and $C_{\alpha\beta\lambda\mu}$, $E_{\alpha\beta\lambda\mu}$ are the elastic compliance, and stiffness tensor respectively. Note that the Einstein summation convention is implicitly assumed throughout the paper.

For isotropic, homogeneous elastic materials, 2D elastic compliance tensor can be expressed as

$$C_{\alpha\beta\lambda\mu} := \frac{1+v}{E}\delta_{\alpha\lambda}\delta_{\beta\mu} - \frac{v}{E}\delta_{\alpha\beta}\delta_{\lambda\mu}; \tag{2.2}$$

and the 2D elastic stiffness modulus tensor has the form

$$E_{\alpha\beta\lambda\mu} := \frac{E}{1+v}\delta_{\alpha\lambda}\delta_{\beta\mu} + \frac{Ev}{1-v^2}\delta_{\alpha\beta}\delta_{\lambda\mu}, \tag{2.3}$$

where E is Young's modulus; and v is Poisson's ratio.

The above elastic stiffness tensor and compliance tensor are only valid in plane stress state. To find elastic stiffness tensor and compliance tensor in plane strain state, one can replace Young's modulus and Poisson's ratio by

$$E \Rightarrow \frac{E}{1-v^2}, \quad v \Rightarrow \frac{v}{1-v}, \tag{2.4}$$

and the corresponding tensors in plane strain state are:

$$C_{\alpha\beta\lambda\mu} = \frac{1+v}{E}\delta_{\alpha\lambda}\delta_{\beta\mu} - \frac{(1+v)v}{E}\delta_{\alpha\beta}\delta_{\lambda\mu}, \tag{2.5}$$

$$E_{\alpha\beta\lambda\mu} = \frac{E}{1+v}\delta_{\alpha\lambda}\delta_{\beta\mu} + \frac{Ev}{(1+v)(1-2v)}\delta_{\alpha\beta}\delta_{\lambda\mu}. \tag{2.6}$$

In the rest of paper, we mainly deal with the plane stress description, with the understanding that all the results are valid for plane strain as well, unless it is indicated otherwise.

In absence of body force, one may introduce the Airy stress function, such that

$$\sigma_{\alpha\beta} = \epsilon_{\alpha\lambda}\epsilon_{\beta\mu}\varphi_{,\lambda\mu}, \quad \text{or} \quad \varphi_{,\alpha\beta} = \epsilon_{\alpha\lambda}\epsilon_{\beta\mu}\sigma_{\lambda\mu}, \tag{2.7}$$

where $\epsilon_{\alpha\beta}$ are the 2D permutation tensor, i.e.

$$[\epsilon_{\alpha\beta}] = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{2.8}$$

An useful identity of the 2D permutation tensor is self-contraction,

$$\epsilon_{\gamma\alpha}\epsilon_{\gamma\beta} = \delta_{\alpha\beta}. \tag{2.9}$$

For plane stress state, the density of the complementary energy is

$$L_c(\boldsymbol{\sigma};\mathbf{x}) = \frac{1}{2E}[(1+v)\sigma_{\alpha\beta}\sigma_{\alpha\beta} - v\sigma_{\lambda\lambda}\sigma_{\mu\mu}]. \tag{2.10}$$

In terms of the Airy stress function, the complementary energy density in plane stress state can be written as

$$L_c^{(se)}(\partial^2\varphi;\mathbf{x}) = \frac{1}{2E}[(1+v)\varphi_{,\alpha\beta}\varphi_{,\alpha\beta} - v\varphi_{,\lambda\lambda}\varphi_{,\mu\mu}], \tag{2.11}$$

where $\partial^2\varphi = \{\varphi_{,\alpha\beta}\}$, α, $\beta = 1$, 2, and the complementary energy density in plane strain state is

$$L_c^{(sa)}(\partial^2\varphi;\mathbf{x}) = \frac{(1+v)}{2E}[\varphi_{,\alpha\beta}\varphi_{,\alpha\beta} - v\varphi_{,\lambda\lambda}\varphi_{,\mu\mu}]. \tag{2.12}$$

Assume that the stress function is prescribed over the whole boundary. The total complementary potential energy is

$$\Pi_c(\varphi;\mathbf{x}) = \int\int_\Omega L_c(\partial^2\varphi;\mathbf{x})\,\mathrm{d}\Omega. \tag{2.13}$$

The Euler–Lagrangian Equation of the complementary energy functional is the biharmonic equation, i.e.

$$\frac{\partial L_c}{\partial\varphi} - \frac{\partial}{\partial x_\alpha}\frac{\partial L_c}{\partial\varphi_{,\alpha}} + \frac{\partial^2}{\partial x_\alpha \partial x_\beta}\frac{\partial L_c}{\partial\varphi_{,\alpha\beta}} = 0 \Rightarrow \varphi_{,\alpha\alpha\beta\beta} = 0, \tag{2.14}$$

which carries different information other than the 2D Navier equations. In specific, the biharmonic equation satisfied by stress function, φ, characterizes the compatibility constraint of 2D elasticity.

Given linear operator $\mathbf{L} = L_{\alpha\beta\lambda\mu}\mathbf{e}_\alpha \otimes \mathbf{e}_\beta \otimes \mathbf{e}_\lambda \otimes \mathbf{e}_\mu$; and $\mathbf{a} = a_\alpha\mathbf{e}_\alpha \neq 0$, $\mathbf{b} = b_\alpha\mathbf{e}_\alpha \neq 0$; $\mathbf{a},\mathbf{b} \in \mathbb{R}^2$. Let $\mathbf{A} := \mathbf{a}\otimes\mathbf{b}$. We say that $\mathbf{L}$ is strongly elliptic if

$$\mathbf{A}:\mathbf{L}:\mathbf{A} = L_{\alpha\beta\lambda\mu}a_\alpha b_\beta a_\lambda b_\mu > 0. \tag{2.15}$$

It is not difficult to verify the positive definite condition for complementary potential energy,

$$\frac{1+v}{E} > 0, \quad \frac{1-2v}{E} > 0, \quad \text{(plane stress)}; \tag{2.16}$$

$$\frac{1+v}{E} > 0, \quad \frac{(1+v)(1-2v)}{E} > 0, \quad \text{(plane strain)}. \tag{2.17}$$

Therefore for a two-dimensional elastic solid, elastic compliance tensor being positive definite is equivalent to $E > 0$ and $-1 < v < 1/2$, which is the same as for three dimensional elastic solids (see [13]).

2.2. One parameter group of invariant transformation

Lie group analysis of partial differential equations has been a triumph in mathematics, physics, and engineering science. For contemporary expositions, readers may consult monographies by Ibragimov [19,20], Olver [27]. In this section, we shall briefly summarize the main technical ingredients of generalized symmetry transformation, or Lie–Bäcklund transformation, and Noether theorem. The notation adopted in this paper mainly follows Olver [27] and Ibragimov [20]. In this paper, we are only interested in the partial differential equation of a scalar function and its associated variational problem.

Let

$$\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n, \tag{2.18}$$

where $\mathbb{R}^n$ is the n-dimensional Cartesian space and

$$\partial u = \left\{ \frac{\partial u}{\partial x_i} \right\}, \ldots, \partial^s u = \left\{ \frac{\partial^s u}{\partial x_i^s} \right\}, \ldots, \quad 1 < s, \ \ 1 \leqslant i \leqslant n. \tag{2.19}$$

The space $\mathbf{Z}$ is a direct product,

$$\mathbf{Z} = \mathbb{R}^n \times V, \tag{2.20}$$

where V is an infinite dimensional vector space with component

$$\mathbf{y} = (u, \partial u, \ldots, \partial^s u, \ldots) \in V, \tag{2.21}$$

The point $\mathbf{z} = (z_1, z_2, \ldots) \in \mathbf{Z}$ can be written as

$$\mathbf{z} = (\mathbf{x}, \mathbf{y}) = (x_1, x_2, \ldots, x_n, u, \partial u, \ldots, \partial^s u, \ldots). \tag{2.22}$$

Denote the vector space of all differential functions of finite order m as $\mathscr{A}$ and any finite sequence of $\mathbf{z}$ as $[\mathbf{z}]$. Then elements of $\mathscr{A}$ may be written as $f([\mathbf{z}]) \in \mathscr{A}$.

Consider a formal one-parameter Lie group G of generalized transformation of following type,

$$x_i^* = \exp(\epsilon \xi_i) x_i; \tag{2.23}$$

$$u^* = \exp(\epsilon \eta) u; \tag{2.24}$$

where

$$\xi_i := \left. \frac{\mathrm{d}x_i^*}{\mathrm{d}\epsilon} \right|_{\epsilon=0}, \tag{2.25}$$

$$\eta := \left. \frac{\mathrm{d}u^*}{\mathrm{d}\epsilon} \right|_{\epsilon=0}. \tag{2.26}$$

The transformation is generalized in a sense that its infinitesimal generators have the form

$$\xi_i = \xi_i(\mathbf{x}, u, \partial u, \ldots, \partial^s u, \ldots) = \xi_i(\mathbf{z}), \quad 1 \leqslant i \leqslant n \tag{2.27}$$

$$\eta = \eta(\mathbf{x}, u, \partial u, \ldots, \partial^s u, \ldots) = \eta(\mathbf{z}). \tag{2.28}$$

Definition 2.1 [27]. A generalized vector field is a formal expression of the following form

$$\mathbf{v} = \xi_i(\mathbf{z}) \frac{\partial}{\partial x_i} + \eta(\mathbf{z}) \frac{\partial}{\partial u}. \tag{2.29}$$

Theorem 2.1 (Prolongation Formula [19,27]). *An infinite prolongation* (*or prolongation for short*) *formula of the generalized* (*Lie–Bäcklund*) *vector field* $\mathbf{v}$ *is*

$$\mathrm{pr}\mathbf{v} = \mathbf{v} + \sum_{1 \leqslant s} \eta^{(s)}_{i_1,\ldots,i_s} \frac{\partial}{\partial u_{i_1,\ldots,i_s}}, \tag{2.30}$$

where

$$\eta^{(s)}_{i_1,\ldots,i_s} = D_{i_1}, \ldots, D_{i_s}\left(\eta - \xi_j u_{,j}\right) + \xi_j u_{,ji_1,\ldots,i_s}, \quad s = 1, 2, \ldots \tag{2.31}$$

and

$$D_i := \frac{\partial}{\partial x_i} + u_{,i} \frac{\partial}{\partial u} + \cdots + u_{,ii_1,\ldots,i_{s-1}} \frac{\partial}{\partial u_{,i_1,\ldots,i_s}} + \cdots \tag{2.32}$$

A pth order prolongation formula of generalized vector field $\mathbf{v}$ *is*

$$\mathrm{pr}^{(p)}\mathbf{v} = \mathbf{v} + \sum_{1 \leqslant s \leqslant p} \eta^{(s)}_{i_1,\ldots,i_s} \frac{\partial}{\partial u_{i_1,\ldots,i_s}} \tag{2.33}$$

where p is the maximal order of non-vanishing derivatives, and

$$\eta^{(s)}_{i_1,\ldots,i_s} = D_{i_1}, \ldots, D_{i_s}\left(\eta - \xi_j u_{,j}\right) + \xi_j u_{,ji_1,\ldots,i_s}, \quad 1 \leqslant s \leqslant p \tag{2.34}$$

$$D_i = \frac{\partial}{\partial x_i} + u_{,i} \frac{\partial}{\partial u} + \cdots + u_{,ii_1,\ldots,i_{p-1}} \frac{\partial}{\partial u_{,i_1,\ldots,i_p}}. \tag{2.35}$$

Consider a qth order scaler partial differential equation (PDE) denoted by

$$F = F([\mathbf{z}]) = F(\mathbf{x}, u, \partial u, \ldots, \partial^q u), \tag{2.36}$$

where $q \geqslant 1$ is some positive integer. Define the differential manifold

$$[F] : F = 0, \dots, D_{1_1}, \dots, D_{i_k} F = 0, \quad k = p + q. \tag{2.37}$$

We have the following theorem.

Theorem 2.2 (e.g. [27]). *Let G be a group of the Lie–Bäcklund transformation, with tangent vector field* $\mathrm{pr}\mathbf{v}$. *The differential manifold* $[F]$ *is invariant under G, if and only if*

$$\mathrm{pr}\mathbf{v}F|_{[F]} = 0. \tag{2.38}$$

Note that

$$\mathrm{pr}\mathbf{v}F = 0 \Rightarrow \mathrm{pr}^{(q)}\mathbf{v}F([\mathbf{z}]) = 0. \tag{2.39}$$

Eq. (2.39) *is often referred to as the determining equation.*

Define the Euler–Lagrangian operator

$$E := \frac{\partial}{\partial u} + \sum_{1 \leqslant s} (-1)^s D_{1_1}, \dots, D_{i_s} \frac{\partial}{\partial u_{i_1,\dots,i_s}}; \tag{2.40}$$

and the Noether operator

$$N_i = \xi_i + (\eta - \xi_\ell u_{,\ell}) \left\{ \frac{\partial}{\partial u_{,i}} + \sum_{1 \leqslant s} (-1)^s D_{j_1}, \dots, D_{j_s} \frac{\partial}{\partial u_{,ij_1,\dots,j_s}} \right\}$$
$$+ \sum_{1 \leqslant r} D_{k_1 \cdots k_r} (\eta - \xi_\ell u_{,\ell}) \left\{ \frac{\partial}{\partial u_{,ik_1,\dots,k_r}} + \sum_{s \geqslant 1} (-1)^s D_{j_1}, \dots, D_{j_s} \frac{\partial}{\partial u_{,ik_1,\dots,k_r j_1,\dots,j_s}} \right\}. \tag{2.41}$$

The celebrated Noether theorem (Noether [24]) can be stated as follows.

Theorem 2.3 [18]. *Given a formal Lie–Bäcklund transformation group with Lie–Bäcklund operator* $\mathrm{pr}\mathbf{v}$, *the following identity*

$$\mathrm{pr}\mathbf{v} + D_i \xi_i = (\eta - \xi_j u_{,j}) E + D_i N_i \tag{2.42}$$

holds.

Consequently, for $\mathbf{y}$ satisfying the Euler–Lagrangian equation

$$E(L([\mathbf{z}])) = 0, \tag{2.43}$$

the prolongation equation equals to a divergence form

$$\text{pr}\mathbf{v}(L) + L\text{Div}\,\xi = \text{Div}\,\mathbf{N}(L) = 0. \tag{2.44}$$

Taking into account null Lagrangians, there exist functions $\{B_i(\mathbf{z})\}$ such that

$$\text{pr}\mathbf{v}(L) + L\text{Div}\,\xi = \text{Div}\,\mathbf{B}, \tag{2.45}$$

then the following conservation laws yield

$$\text{Div}\,\mathbf{P}(L) = 0, \tag{2.46}$$

where $P_i(L) = N_i(L) - B_i$, or $\mathbf{P}(L) = \mathbf{N}(L) - \mathbf{B}$.

Remark 2.1. The divergence term, Div**B**, can be taken into consideration if proper boundary conditions are prescribed on the whole boundary of the domain of the variational problem. Otherwise, improper boundary conditions specification may affect the original boundary data, and subsequently change the variational problem (See [6] for detail).

3. Symmetry and invariant group

3.1. Lie–Bäcklund symmetry

For planar elasticity, the Euler–Lagrangian equation of complementary energy potential is a biharmonic equation satisfied by the Airy stress function φ. Since the variational symmetry group is a subgroup of invariant transformation admitted by its Euler–Lagrangian equation, we begin with finding the Lie–Bäcklund symmetry admitted by biharmonic equations. It should be mentioned that the Lie group of point transformation admitted by biharmonic equation has been studied by Bluman & Gregory [1]. Here, we are looking for more general symmetry—the Lie–Bäcklund symmetry admitted by the biharmonic equation.

Let $\mathbf{x} = (x_1, x_2)$ and $u = \varphi(x_1, x_2)$.

$$\mathbf{x}^* = \exp(\epsilon\boldsymbol{\xi})\mathbf{x}, \tag{3.1}$$

$$\varphi^* = \exp(\epsilon\eta)\varphi. \tag{3.2}$$

The generalized vector field is given as

$$\mathbf{v} = \xi_\alpha \frac{\partial}{\partial x^\alpha} + \eta \frac{\partial}{\partial \varphi}. \tag{3.3}$$

Consider the infinitesimal generators of the following forms

$$\xi_\alpha = \xi_\alpha(\mathbf{x}), \tag{3.4}$$

$$\eta = f(\mathbf{x}) + g(\mathbf{x})\varphi + h_\gamma(\mathbf{x})\varphi_{,\gamma} + k_{\lambda\mu}(\mathbf{x})\varphi_{,\lambda\mu} + p_\lambda(\mathbf{x})\varphi_{,\alpha\alpha\lambda}, \tag{3.5}$$

where $f(\mathbf{x})$, $g(\mathbf{x})$ are the unknown scalar functions; $\xi_\alpha(\mathbf{x})$, $h_\alpha(\mathbf{x}), k_{\lambda\mu}(\mathbf{x})$, and $p_\lambda(\mathbf{x})$ are the unknown vector or tensorial functions.

By the Theorem 2.2, the invariant conditions, or the determining equations, are

$$\mathrm{pr}^{(4)}\mathbf{v}\left(\frac{\partial^4\varphi}{\partial x_\alpha\partial x_\alpha\partial x_\beta\partial x_\beta}\right)=0, \tag{3.6}$$

where

$$\mathrm{pr}^{(4)}\mathbf{v}=\xi_\alpha\frac{\partial}{\partial x_\alpha}+\eta\frac{\partial}{\partial\phi}+\eta^{(1)}_\alpha\frac{\partial}{\partial\varphi_{,\alpha}}+\cdots+\eta^{(4)}_{\alpha\beta\lambda\mu}\frac{\partial}{\partial\varphi_{,\alpha\beta\lambda\mu}}. \tag{3.7}$$

This leads to an algebraic equation for the fourth order extensions,

$$\eta^{(4)}_{\alpha\alpha\beta\beta}=0. \tag{3.8}$$

The determining equation can be written as follows,

$$\begin{aligned}\eta^{(4)}_{\alpha\beta\alpha\beta}&=f_{,\alpha\beta\alpha\beta}+\left[g_{,\alpha\beta\alpha\beta}\varphi+4g_{,\alpha\beta\beta}\varphi_{,\alpha}+2g_{,\alpha\alpha}\varphi_{,\beta\beta}+4g_{,\alpha\beta}\varphi_{,\alpha\beta}+4g_{,\alpha}\varphi_{,\alpha\beta\beta}\right]+\left[(h_{\gamma,\alpha\beta\alpha\beta}-\xi_{\gamma,\alpha\beta\alpha\beta})\varphi_{,\gamma}\right.\\&\quad\left.+4(p_{\gamma,\alpha\alpha\beta}-\xi_{\gamma,\alpha\alpha\beta})\varphi_{,\gamma\beta}+4(h_{\gamma,\alpha\beta}-\xi_{\gamma,\alpha\beta})\varphi_{,\gamma\alpha\beta}+2(h_{\gamma,\alpha\alpha}-\xi_{\gamma,\alpha\alpha})\varphi_{,\gamma\beta\beta}+4(h_{\gamma,\alpha}-\xi_{\gamma,\alpha})\varphi_{,\gamma\alpha\beta\beta}\right]\\&\quad+\left[k_{\lambda\mu,\alpha\beta\alpha\beta}\varphi_{,\lambda\mu}+4k_{\lambda\mu,\alpha\alpha\beta}\varphi_{,\lambda\mu\beta}+2k_{\lambda\mu,\alpha\alpha}\varphi_{,\lambda\mu\beta\beta}+4k_{\lambda\mu,\alpha\beta}\varphi_{,\lambda\mu\alpha\beta}+4k_{\lambda\mu,\beta}\varphi_{,\lambda\mu\alpha\beta\beta}+k_{\lambda\mu}\varphi_{,\lambda\mu\alpha\beta\alpha\beta}\right]\\&\quad+\left[p_{\lambda,\alpha\alpha\beta\beta}\varphi_{,\gamma\gamma\lambda}+4p_{\lambda,\alpha\beta\beta}\varphi_{,\gamma\gamma\lambda\alpha}+4p_{\lambda,\alpha\beta}\varphi_{,\gamma\gamma\lambda\alpha\beta}+2p_{\lambda,\alpha\alpha}\varphi_{,\gamma\gamma\lambda\beta\beta}+4p_{\lambda,\alpha}\varphi_{,\gamma\gamma\lambda\beta\beta}+p_\lambda\varphi_{,\gamma\gamma\lambda\alpha\alpha\beta\beta}\right]\\&=0.\end{aligned} \tag{3.9}$$

Let $\bar{\xi}_\kappa:=\xi_\kappa-h_\kappa$. The determining equation can be split into a set of coupled differential equations among unknown functions, $f(\mathbf{x})$, $g(\mathbf{x})$, $h_\kappa(\mathbf{x})$, $k_{\lambda\mu}(\mathbf{x})$, $p_\kappa(\mathbf{x})$, and $\xi_\kappa(\mathbf{x})$:

$$\varphi^0: f_{,\alpha\alpha\beta\beta}=0; \tag{3.10}$$

$$\varphi^1: g_{,\alpha\alpha\beta\beta}=0; \tag{3.11}$$

$$\partial\varphi: 4g_{,\alpha\beta\beta}\varphi_{,\alpha}-\bar{\xi}_{\kappa,\alpha\alpha\beta\beta}\varphi_{,\kappa}=0; \tag{3.12}$$

$$\partial^2\varphi: 4g_{,\alpha\beta}\varphi_{,\alpha\beta}+2g_{,\alpha\alpha}\varphi_{,\beta\beta}-\bar{\xi}_{\kappa,\alpha\beta\beta}\varphi_{,\kappa\alpha}+k_{\lambda\mu,\alpha\alpha\beta\beta}\varphi_{,\lambda\mu}=0; \tag{3.13}$$

$$\partial^3\varphi: 4g_{,\alpha}\varphi_{,\alpha\beta\beta}-4\bar{\xi}_{\kappa,\alpha\beta}\varphi_{,\kappa\alpha\beta}-2\bar{\xi}_{\kappa,\alpha\alpha}\varphi_{,\kappa\beta\beta}+4k_{\lambda\mu,\alpha\beta\beta}\varphi_{,\lambda\mu\alpha}+p_{\kappa,\alpha\alpha\beta\beta}\varphi_{,\kappa\delta\delta}=0; \tag{3.14}$$

$$\partial^4\varphi: -4\bar{\xi}_{\kappa,\alpha}\varphi_{,\kappa\alpha\beta\beta}+4k_{\lambda\mu,\alpha\beta}\varphi_{,\lambda\mu}\varphi_{,\lambda\mu\alpha\beta}+2k_{\lambda\mu,\alpha\alpha}\varphi_{,\lambda\mu\beta\beta}+4p_{\kappa,\alpha\beta\beta}\varphi_{,\kappa\alpha\delta\delta}=0; \tag{3.15}$$

$$\partial^5\varphi: 4k_{\lambda\mu,\alpha}\varphi_{,\lambda\mu\alpha\beta\beta}+4p_{\kappa,\alpha\beta}\varphi_{,\kappa\alpha\beta\mu\mu}=0. \tag{3.16}$$

A set of special solutions of above differential equations are obtained:

$$p_\kappa(\mathbf{x})=p^{(2)}_\kappa x_\alpha x_\alpha+p^{(1)}\epsilon_{\kappa\gamma}x_\gamma+p^{(0)}_\kappa, \tag{3.17}$$

$$k_{\lambda\mu}(\mathbf{x})=k^{(2)}\delta_{\lambda\mu}x_\alpha x_\alpha+(k^{(1)}_\lambda x_\mu+k^{(1)}_\mu x_\lambda)+k^{(0)}_{\lambda\mu}, \tag{3.18}$$

$$\xi_\alpha(\mathbf{x}) = P^1_{\alpha\lambda\mu}x_\lambda x_\mu + \theta^1_{\alpha\beta}x_\beta + d^1_\alpha, \tag{3.19}$$

$$h_\alpha(\mathbf{x}) = P^2_{\alpha\lambda\mu}x_\lambda x_\mu + \theta^2_{\alpha\beta}x_\beta + d^2_\alpha. \tag{3.20}$$

The superscript, (i) in a coefficient indicates the order of polynomial that the coefficient proceeds. The superscript, $\alpha = 1, 2$, indicates different sets of coefficients. The free parameter tensors $P^\imath_{\alpha\lambda\mu}$ and $\theta^\imath_{\alpha\beta}$ satisfy the conditions

$$P^\imath_{\alpha\lambda\mu} = P^\imath_{\alpha\mu\lambda}, \quad \imath = 1,2 \tag{3.21}$$

$$P^\imath_{\alpha\lambda\mu} = -P^\imath_{\lambda\alpha\mu}, \quad \alpha \neq \lambda, \ \imath = 1,2 \tag{3.22}$$

$$\theta^\imath_{\alpha\beta}\theta^\imath_{\beta\gamma} = \delta_{\alpha\gamma}\det|\theta^\imath|, \quad \text{and} \quad \det|\theta^\imath| = \theta^\imath_{11}\theta^\imath_{22} - \theta^\imath_{12}\theta^\imath_{21}, \quad \imath = 1,2. \tag{3.23}$$

Denote $\Delta P_{\alpha\lambda\mu} := P^1_{\alpha\lambda\mu}, -P^2_{\alpha\lambda\mu}$, $\Delta\theta_{\alpha\beta} := \theta^1_{\alpha\beta} - \theta^2_{\alpha\beta}$, $\Delta d_\alpha = d^1_\alpha - d^2_\alpha$. Then

$$\begin{aligned} \bar{\xi}_\alpha(\mathbf{x}) &= \Delta P_{\alpha\lambda\mu}x_\lambda x_\mu + \Delta\theta_{\alpha\beta}x_\beta + \Delta d_\alpha \\ g(\mathbf{x}) &= \frac{1}{2}\bar{\xi}_{\alpha,\alpha} + c = \Delta P_{\alpha\alpha\mu}x_\mu + \frac{1}{2}\Delta\theta_{\alpha\alpha} + c \end{aligned} \tag{3.24}$$

$$f(\mathbf{x}) = f(x), \tag{3.25}$$

where $f(\mathbf{x})$ can be any function that satisfies biharmonic equation, i.e. $f_{,\alpha\alpha\beta\beta} = 0$.

3.2. Variational symmetry

The symmetry group admitted by the Euler–Lagrangian equation may not necessarily yield variational symmetry. A simple procedure to find variational symmetry group is to test all invariant solutions admitted by the Euler–Lagrangian equation and to select those that indeed satisfy both determining equation and prolongation equation, i.e.

$$\begin{aligned} &\mathrm{pr}\mathbf{v}L_c + (D_\alpha\xi_\alpha)L_c = \eta^{(2)}_{\alpha\beta}\frac{\partial L_c}{\partial\varphi_{,\alpha\beta}} + (D_\alpha\xi_\alpha)L_c = 0 \\ &\quad\Rightarrow \left(D_\alpha D_\beta(\eta - \xi_{,\gamma}\varphi_{,\gamma}) + \xi_\gamma\varphi_{,\gamma\alpha\beta}\right)\frac{\partial L_c}{\partial\varphi_{,\alpha\beta}} + \xi_{\gamma,\gamma}L_c \\ &\quad= \frac{\partial L_c}{\partial\varphi_{,\alpha\beta}}\Big\{f_{,\alpha\beta}. + \left(g_{,\alpha\beta}\varphi + g_{,\alpha}\varphi_{,\beta} + g_{,\beta}\varphi_{,\alpha} + g\varphi_{,\alpha\beta}\right) \\ &\qquad + h_\kappa\varphi_{,\kappa\alpha\beta} - \left(\bar{\xi}_{\kappa,\alpha\beta}\varphi_{,\kappa} + \bar{\xi}_{\kappa,\alpha}\varphi_{,\kappa\beta} + \bar{\xi}_{\kappa,\beta}\varphi_{,\kappa\alpha}\right) \\ &\qquad + \left(+ k_{\lambda\mu,\alpha}\varphi_{,\lambda\mu\beta} + k_{\lambda\mu,\beta}\varphi_{,\lambda\mu\alpha} + k_{\lambda\mu}\varphi_{,\lambda\mu\alpha\beta}\right) \\ &\qquad + \left(p_{\kappa,\alpha\beta}\varphi_{,\kappa\gamma\gamma} + p_{\kappa,\alpha}\varphi_{,\kappa\beta\gamma\gamma} + p_{\kappa,\beta}\varphi_{,\kappa\alpha\gamma\gamma} + p_\kappa\varphi_{,\kappa\alpha\beta\gamma\gamma}\right)\Big\} \end{aligned} \tag{3.26}$$

$$\qquad + \frac{1}{2E}\left((1+\nu)\varphi_{,\alpha\alpha}\varphi_{,\alpha\alpha} - \nu\varphi_{,\alpha\alpha}\varphi_{,\beta\beta}\right)\xi_{\gamma,\gamma} = 0, \tag{3.27}$$

where

$$\frac{\partial L_c}{\partial \varphi_{,\alpha\beta}} = \left(\frac{1+v}{E} \varphi_{,\alpha\beta} - \frac{v}{E} \varphi_{,\tau\tau} \delta_{\alpha\beta} \right). \tag{3.28}$$

Consequently, it yields the following Killing's equations,

$$\partial^2 \varphi : \frac{\partial L_c}{\partial \varphi_{,\alpha\beta}} f_{,\alpha\beta} = 0; \tag{3.29}$$

$$\varphi \partial^2 \varphi : \frac{\partial L_c}{\partial \varphi_{,\alpha\beta}} (\varphi g_{,\alpha\beta}) = 0; \tag{3.30}$$

$$\partial \varphi \partial^2 \varphi : \frac{\partial L_c}{\partial \varphi_{,\alpha\beta}} \left(g_{,\alpha} \varphi_{,\beta} + g_{,\beta} \varphi_{,\alpha} - \bar{\xi}_{\kappa,\alpha\beta} \varphi_{\kappa} \right) = 0; \tag{3.31}$$

$$\begin{aligned} &\partial^2 \varphi \partial^2 \varphi : \frac{\partial L_c}{\partial \varphi_{,\alpha\beta}} \left(g \varphi_{,\alpha\beta} - \bar{\xi}_{\kappa,\alpha} \varphi_{,\kappa\beta} - \bar{\xi}_{\kappa,\beta} \varphi_{,\kappa\alpha} + \frac{1}{2} \xi_{\gamma,\gamma} \varphi_{,\alpha\beta} \right) = 0; \\ &\partial^2 \varphi \partial^3 \varphi : \frac{\partial L_c}{\partial \varphi_{,\alpha\beta}} \left(h_{\kappa} \varphi_{,\kappa\alpha\beta} + k_{\lambda\mu,\alpha} \varphi_{,\lambda\mu\beta} + k_{\lambda\mu,\beta} \varphi_{,\lambda\mu\alpha} + p_{\kappa,\alpha\beta} \varphi_{,\kappa\gamma\gamma} \right) = 0; \end{aligned} \tag{3.32}$$

$$\partial^2 \varphi \partial^4 \varphi : \frac{\partial L_c}{\partial \varphi_{,\alpha\beta}} \left(k_{\lambda\mu} \varphi_{,\lambda\mu\alpha\beta} + p_{\lambda,\alpha} \varphi_{,\lambda\beta\gamma\gamma} + p_{\lambda,\beta} \varphi_{,\lambda\alpha\gamma\gamma} \right) = 0; \tag{3.33}$$

$$\partial^2 \varphi \partial^5 \varphi : \frac{\partial L_c}{\partial \varphi_{,\alpha\beta}} p_{\kappa} \varphi_{,\kappa\alpha\beta\mu\mu} = 0. \tag{3.34}$$

By substituting solutions (3.17)–(3.20) and (3.20)–(3.25) into the above equations, one may find the following additional constraints:

$$\frac{\partial^2 f}{\partial x_\alpha \partial x_\beta} = 0, \tag{3.35}$$

$$h_\alpha = 0, \tag{3.36}$$

$$k_{\alpha\beta} = 0, \tag{3.37}$$

$$p_\alpha = 0. \tag{3.38}$$

Thus, the variational invariant transformations are [1]

$$\xi_1 = a_1 x_1 - a_2 x_2 + a_3; \tag{3.39}$$

$$\xi_2 = a_2 x_1 + a_1 x_2 + a_4; \tag{3.40}$$

[1] There may exist some higher order variational symmetry.

$$\eta = (a_5 x_1 + a_6 x_2 + a_7) + a_1 \varphi; \tag{3.41}$$

where a_i are arbitrary constants. Note that Eqs. (3.39) and (3.40) may be written as $\xi_\alpha = a_1 x_\alpha + a_2 \epsilon_{\alpha\beta} x_\beta + b_\alpha$, with $b_1 = a_3$ and $b_2 = a_4$.

When $v = 1/2$ for plane strain case, we have additional inversion transformations

$$\xi_1 = a_8(x_1^2 - x_2^2) + 2a_9 x_1 x_2, \tag{3.42}$$

$$\xi_2 = 2a_8 x_1 x_2 - a_9(x_1^2 - x_2^2), \tag{3.43}$$

$$\eta = (2a_8 x_1 - 2a_9 x_2)\varphi. \tag{3.44}$$

Note that similar solution can be obtained when $v = 1$ for plane stress. Since Poisson's ratio can not be greater than 1/2, invariant solution at $v = 1$ is not realistic.

Therefore, the Lie group of variational invariant transformation, i.e., the tangent vector fields, are

$$\begin{aligned}
&X_1 = -x_2 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2};\\
&X_2 = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + \varphi \frac{\partial}{\partial \varphi};\\
&X_3 = \frac{\partial}{\partial x_1}; \quad X_4 = \frac{\partial}{\partial x_2};\\
&X_5 = x_1 \frac{\partial}{\partial \varphi}; \quad X_6 = x_2 \frac{\partial}{\partial \varphi}; \quad X_7 = \frac{\partial}{\partial \varphi}.
\end{aligned} \tag{3.45}$$

When $v = 1/2$, there are two additional invariant vector fields for plane strain state

$$\begin{aligned}
&X_8 = (x_1^2 - x_2^2) \frac{\partial}{\partial x_1} + 2x_1 x_2 \frac{\partial}{\partial x_2} + 2x_1 \varphi \frac{\partial}{\partial \varphi},\\
&X_9 = -2x_1 x_2 \frac{\partial}{\partial x_1} + (x_1^2 - x_2^2) \frac{\partial}{\partial x_2} - 2x_2 \varphi \frac{\partial}{\partial \varphi}.
\end{aligned}$$

Remark 3.1. It would be interesting to compare the current results with the variational symmetry group of the Navier equation. For the Navier equation, the inversion symmetry occurs when $7\mu + 3\lambda = 0$, i.e. $v = 7/8$ [26]. The inversion symmetry discovered in this paper is realistic, because it is right at incompressible limit.

3.3. Divergence symmetry

The variational symmetry group found is only a subgroup of point transformation as shown in (3.45). However, the generalized transformations, or the Lie–Bäcklund transformations, can have divergence symmetry. There exist functions, B_α, such that the Noether theorem holds in the following the Bessel–Hagen form,

$$\mathbf{prv}(L_c) + (D_\alpha \xi_\alpha) L_c = D_\alpha B_\alpha. \tag{3.46}$$

In the following, several divergence symmetric transformations are found.

3.3.1. Divergence symmetry I

Consider the following proper Lie–Bäcklund transformation, which is admitted by the biharmonic equation,

$$\xi_\alpha = 0, \quad \text{and} \quad \eta = \psi(x_1, x_2), \tag{3.47}$$

where the function ψ is an arbitrary solution of biharmonic equation. Choose

$$B_\alpha = h_\beta \vartheta_{,\gamma\gamma\alpha} \varphi_{,\beta}, \tag{3.48}$$

where h_β is a constant vector and ϑ is another solution of biharmonic equation, i.e. $\vartheta_{,\alpha\alpha\beta\beta} = 0$. Consequently,

$$D_\alpha B_\alpha = h_\beta \vartheta_{,\gamma\gamma\alpha} \varphi_{,\alpha\beta}. \tag{3.49}$$

Hence, by Noether identity (2.42), one may find that

$$\psi_{,\alpha\beta} = E_{\alpha\beta\lambda\mu} h_\mu \vartheta_{,\gamma\gamma\lambda}, \tag{3.50}$$

or vice versa,

$$h_\beta \vartheta_{,\gamma\gamma\alpha} = C_{\alpha\beta\lambda\mu} \psi_{,\lambda\mu}, \tag{3.51}$$

$$B_\alpha = C_{\alpha\beta\lambda\mu} \psi_{,\lambda\mu} \varphi_{,\beta}, \tag{3.52}$$

where $E_{\alpha\beta\lambda\mu}$ and $C_{\alpha\beta\lambda\mu}$ are the elastic stiffness tensor and elastic compliance tensor defined in Eqs. (2.2) and (2.3).

It is worth verifying that indeed,

$$\psi_{,\alpha\alpha\beta\beta} = h_\mu \left(\frac{E}{1+\nu} + \frac{2E\nu}{1-\nu^2} \right) \vartheta_{,\gamma\gamma\mu\beta\beta} = 0. \tag{3.53}$$

A tangent vector field with divergence symmetry is found to be

$$X_{\rm I} = \psi(x_1, x_2) \frac{\partial}{\partial \varphi}; \tag{3.54}$$

with $\psi_{,\alpha\alpha\beta\beta} = 0$.

3.3.2. Divergence symmetry II

Let

$$\xi_\alpha = 0, \quad \text{and} \quad \eta = b_{\mathrm{II}\gamma}\varphi_{,\gamma} \tag{3.55}$$

where $\{b_{\mathrm{II}\gamma}\}$ is a constant vector. It can be readily shown that

$$\mathrm{pr}\mathbf{v}(L_c) + (D_\alpha\xi_\alpha)L_c = \eta^{(2)}_{\alpha\beta}\frac{\partial L_c}{\partial\varphi_{,\alpha\beta}} = b_{\mathrm{II}\gamma}\frac{\partial}{\partial x_\gamma}\left\{C_{\alpha\beta\lambda\mu}\varphi_{,\alpha\beta}\varphi_{,\lambda\mu}\right\}. \tag{3.56}$$

Choose

$$B_\alpha = \frac{1}{2}b_{\mathrm{II}\alpha}C_{\lambda\mu\gamma\delta}\varphi_{\lambda\mu}\varphi_{\gamma\delta}. \tag{3.57}$$

The divergence symmetric tangent vector field is

$$X_{\mathrm{II}1} = \varphi_{,1}\frac{\partial}{\partial\varphi}, \tag{3.58}$$

$$X_{\mathrm{II}2} = \varphi_{,2}\frac{\partial}{\partial\varphi}. \tag{3.59}$$

This divergence symmetry is equivalent to the variational symmetry due to coordinate translation.

3.3.3. Divergence symmetry III

Let

$$\xi_\alpha = 0, \quad \text{and} \quad \eta = b_{\mathrm{III}}(\varphi - x_\gamma\varphi_{,\gamma}), \tag{3.60}$$

where b_{III} is an arbitrary constant. It is straightforward to verify that

$$D_\alpha D_\beta\eta = -b_{\mathrm{III}}(\varphi_{,\alpha\beta} + x_\gamma\varphi_{,\gamma\alpha\beta}) \tag{3.61}$$

$$\mathrm{pr}\mathbf{v}(L_c) + (D_\alpha\xi_\alpha)L_c = -\frac{b_{\mathrm{III}}}{2}\frac{\partial}{\partial x_\gamma}\left\{C_{\alpha\beta\lambda\mu}\varphi_{,\alpha\beta}\varphi_{,\lambda\mu}x_\gamma\right\} \tag{3.62}$$

Choose $B_\alpha = -a_{\mathrm{II}}L_c x_\gamma$. The following vector field is divergence symmetric,

$$X_{\mathrm{III}} = (\varphi - x_\gamma\varphi_{,\gamma})\frac{\partial}{\partial\varphi}. \tag{3.63}$$

This divergence symmetry is equivalent to the variational symmetry due to scaling.

3.3.4. Divergence symmetry IV

Let

$$\xi_\alpha = 0, \quad \text{and} \quad \eta = b_{\mathrm{IV}}\epsilon_{\lambda\mu}x_\lambda\varphi_{,\mu}. \tag{3.64}$$

One may find that

$$\eta^{(2)}_{\alpha\beta}\frac{\partial L_c}{\partial\varphi_{,\alpha\beta}} = b_{\rm IV}\left(\frac{1+\nu}{E}\varphi_{,\alpha\beta} - \frac{\nu}{E}\varphi_{,\gamma\gamma}\delta_{\alpha\beta}\right)(\epsilon_{\lambda\mu}x_\lambda\varphi_{,\mu\alpha\beta})$$
$$= \frac{b_{\rm IV}}{2}\frac{\rm d}{{\rm d}x_\mu}\left\{\frac{1+\nu}{E}\left(\epsilon_{\lambda\mu}x_\lambda\varphi_{,\alpha\beta}\varphi_{,\alpha\beta}\right) - \frac{\nu}{E}\left(\epsilon_{\lambda\mu}x_\lambda\varphi^2_{,\gamma\gamma}\right)\right\}. \tag{3.65}$$

Choose

$$B_\alpha = -\frac{b_{\rm IV}}{2}\left\{\frac{1+\nu}{E}\left(\epsilon_{\alpha\beta}x_\beta\varphi_{,\lambda\mu}\varphi_{,\lambda\mu}\right) + \frac{\nu}{E}\left(\epsilon_{\alpha\beta}x_\beta\varphi^2_{,\gamma\gamma}\right)\right\}. \tag{3.66}$$

The divergence symmetric tangent vector field is

$$X_{\rm IV} = \epsilon_{\alpha\beta}x_\beta\varphi_{,\alpha}\frac{\partial}{\partial\varphi}. \tag{3.67}$$

Again, this vector field belongs to the equivalent class of a variational symmetry due to coordinate rotation.

3.3.5. Divergence symmetry V

Let

$$\xi_\alpha = 0, \quad \text{and} \quad \eta = b_{\rm V}\varphi_{,\lambda\lambda}. \tag{3.68}$$

Subsequently

$$\eta^{(2)}_{\alpha\beta} = b_{\rm V}\varphi_{,\lambda\lambda\alpha\beta}, \tag{3.69}$$

and

$$\eta^{(2)}_{\alpha\beta}\frac{\partial L_c}{\varphi_{,\alpha\beta}} = b_{\rm V}\frac{1+\nu}{E}\frac{\partial}{\partial x_\alpha}\left(\varphi_{,\beta}\varphi_{,\lambda\lambda\alpha\beta}\right). \tag{3.70}$$

Choose

$$B_\alpha = b_{\rm V}\frac{1+\nu}{E}\varphi_{,\lambda\lambda\alpha\beta}\varphi_{,\beta}. \tag{3.71}$$

The divergence symmetric tangent vector field is

$$X_{\rm V} = \varphi_{,\lambda\lambda}\frac{\partial}{\partial\varphi} \tag{3.72}$$

3.3.6. Divergence symmetry VI

Let

$$\xi_\alpha = 0, \quad \text{and} \quad \eta = b_{\mathrm{VI}\lambda}\varphi_{,\lambda\mu\mu}, \tag{3.73}$$

where $\{b_{\mathrm{VI}\lambda}\}$ is a constant vector. Hence

$$\eta^{(2)}_{\alpha\beta} = b_{\mathrm{VI}\lambda}\varphi_{,\lambda\mu\mu\alpha\beta}. \tag{3.74}$$

Consequently,

$$\eta^{(2)}_{\alpha\beta}\frac{\partial L_c}{\varphi_{,\alpha\beta}} = b_{\mathrm{VI}\lambda}\frac{1+v}{E}\frac{\partial}{\partial x_\alpha}\left(\varphi_{,\beta}\varphi_{,\lambda\mu\mu\alpha\beta}\right). \tag{3.75}$$

Choose

$$B_\alpha = b_{\mathrm{VI}\lambda}\frac{1+v}{E}\varphi_{,\beta}\varphi_{,\lambda\mu\mu\alpha\beta}. \tag{3.76}$$

The corresponding divergence invariant vector field are

$$X_{\mathrm{VI1}} = \varphi_{,1\mu\mu}\frac{\partial}{\partial\varphi}, \tag{3.77}$$

$$X_{\mathrm{VI2}} = \varphi_{,2\mu\mu}\frac{\partial}{\partial\varphi}. \tag{3.78}$$

3.3.7. Divergence symmetry VII

Let

$$\xi_\alpha = 0, \quad \text{and} \quad \eta = b_{\mathrm{VII}}\epsilon_{\kappa\gamma}x_\gamma\varphi_{,\kappa\mu\mu}. \tag{3.79}$$

It can be readily shown that

$$\eta^{(2)}_{\alpha\beta}\frac{\partial L_c}{\varphi_{,\alpha\beta}} = b_{\mathrm{VII}}\frac{1+v}{E}\frac{\partial}{\partial x_\alpha}\left\{2\left[\epsilon_{\kappa\beta}\varphi_{,\mu\mu\kappa\alpha}\varphi_{,\beta}\right] + \left[\epsilon_{\kappa\gamma}x_\gamma\varphi_{,\lambda\mu\mu\alpha\beta}\varphi_{,\beta}\right]\right\}. \tag{3.80}$$

Choose

$$B_\alpha = b_{\mathrm{VII}}\frac{1+v}{E}\left\{2\left[\epsilon_{\kappa\beta}\varphi_{,\mu\mu\kappa\alpha}\varphi_{,\beta}\right] + \left[\epsilon_{\kappa\gamma}x_\gamma\varphi_{,\lambda\mu\mu\alpha\beta}\varphi_{,\beta}\right]\right\}. \tag{3.81}$$

The divergence invariant vector field is

$$X_{\rm VII} = \epsilon_{\kappa\gamma} x_\gamma \varphi_{,\kappa\mu\mu} \frac{\partial}{\partial \varphi}. \tag{3.82}$$

Remark 3.2

- The above calculation is demonstrative; we have not exhausted, in any way, the possibilities of divergence symmetry.
- There could be a confusion regarding the relationship between null Lagrangian and natural boundary conditions. In our problem, prescribed stress function on the boundary will mean the Dirichlet boundary condition, even though it is stress type of boundary condition in physics. In other words, there is a difference between the Nuemann boundary condition in mathematics and "natural boundary condition" in physics.

4. Dual conservation laws

There are two groups of dual conservation laws:

- The genuine variational-symmetric conservation laws:

$$P_\alpha^{(\rm var)} = L_c \xi_\alpha - \frac{1}{E}(\eta - \xi_\gamma \varphi_{,\gamma})\varphi_{,\beta\beta\alpha} + D_\beta(\eta - \xi_\gamma \varphi_{,\gamma}) \frac{\partial L_c}{\partial \varphi_{,\alpha\beta}}; \tag{4.1}$$

- The generalized (divergence-symmetric) conservation laws:

$$P_\alpha^{(\rm div)} = L_c \xi_\alpha - \frac{1}{E}(\eta - \xi_\gamma \varphi_{,\gamma})\varphi_{,\beta\beta\alpha} + D_\beta(\eta - \xi_\gamma \varphi_{,\gamma}) \frac{\partial L_c}{\partial \varphi_{,\alpha\beta}} - B_\alpha. \tag{4.2}$$

4.1. Variational-invariant conservation laws

Variational dual conservation laws in planar elasticity are listed as follows.

(1) Scaling.

Let $a_1 = 1$, and $a_i = 0, i \neq 1$ in variational symmetric transformations (3.39)–(3.44). One then has

$$\xi_\alpha = x_\alpha, \quad \eta = \varphi; \tag{4.3}$$

$$\xi_{\alpha,\beta} = \delta_{\alpha\beta}; \quad \eta_\beta = \varphi_{,\beta}. \tag{4.4}$$

Denote

$$\mathscr{S}_\alpha := P_\alpha^{(\rm var)}\big|_{\{a_1 \neq 0\}}. \tag{4.5}$$

One may find that

$$\mathscr{S}_\alpha = L_c x_\alpha - \frac{1}{E}(\varphi - x_\lambda \varphi_{,\lambda})\varphi_{,\beta\beta\alpha} - x_\gamma \varphi_{,\gamma\beta}\left(\frac{1+v}{E}\varphi_{,\alpha\beta} - \frac{v}{E}\varphi_{,\tau\tau}\delta_{\alpha\beta}\right). \tag{4.6}$$

(2) Coordinate rotation.
Let $a_2 = 1$ and $a_i = 0 (i \neq 2)$ in Eqs. (3.39)–(3.44). One has

$$\xi_\alpha = \epsilon_{\alpha\beta} x_\beta, \quad \eta = 0, \tag{4.7}$$

$$\eta - \xi_\gamma \varphi_{,\gamma} = -\epsilon_{\gamma\lambda} x_\lambda \varphi_{,\gamma}, \quad D_\beta(\eta - \xi_\gamma \varphi_{,\gamma}) = -\epsilon_{\gamma\beta}\varphi_{,\gamma} - \epsilon_{\gamma\lambda} x_\lambda \varphi_{,\gamma\beta}. \tag{4.8}$$

Denote

$$\mathscr{R}_\alpha := P_\alpha^{(\mathrm{var})}\big|_{\{a_2 \neq 0\}}. \tag{4.9}$$

One may find that

$$\mathscr{R}_\alpha = L_c \epsilon_{\alpha\beta} x_\beta + \frac{1}{E}\epsilon_{\gamma\lambda} x_\lambda \varphi_{,\gamma}\varphi_{,\beta\beta\alpha} - (\epsilon_{\gamma\beta}\varphi_{,\gamma} + \epsilon_{\gamma\lambda} x_\lambda \varphi_{,\gamma\beta})\left(\frac{1+v}{E}\varphi_{,\alpha\beta} - \frac{v}{E}\varphi_{,\tau\tau}\delta_{\alpha\beta}\right). \tag{4.10}$$

(3) Coordinate translation.
Let $a_3 = \delta_{1\kappa}, a_4 = \delta_{2\kappa}$, and $a_i = 0, i \neq 3, 4$ in transformations (3.39)–(3.44). Consequently,

$$\xi_\alpha = \delta_{\alpha\kappa}, \quad \eta = 0. \tag{4.11}$$

Define

$$\mathscr{T}_{\alpha(\kappa)} := P_\alpha^{(\mathrm{var})}\big|_{\{a_3, a_4 \neq 0\}}. \tag{4.12}$$

It follows that

$$\mathscr{T}_{\alpha(\kappa)} = L_c \delta_{\alpha\kappa} + \frac{1}{E}\varphi_{,\kappa}\varphi_{,\beta\beta\alpha} - \varphi_{,\kappa\beta}\left(\frac{1+v}{E}\varphi_{,\alpha\beta} - \frac{v}{E}\varphi_{,\tau\tau}\delta_{\alpha\beta}\right), \tag{4.13}$$

where α, $\kappa = 1$, 2.

Remark 4.1. The divergence-free second order tensor $\mathscr{T}_{\alpha(\kappa)}$ may be called as the dual-Eshelby tensor. The following integral

$$J_\kappa^* = \int_{\partial\Omega}\left(L_c n_\kappa + \frac{1}{E}\varphi_{,\kappa}\varphi_{,\beta\beta\alpha} n_\alpha - \varphi_{,\kappa\beta}\left(\frac{1+v}{E}\varphi_{,\alpha\beta} - \frac{v}{E}\varphi_{,\tau\tau}\delta_{\alpha\beta}\right) n_\alpha\right)\mathrm{d}S, \tag{4.14}$$

may be called as dual J integral.

(4) Compatibility Identities.

Let $a_5 = a_6 = 1$ and $a_i = 0$, $i \neq 5, 6$. We may assume that

$$\xi_\alpha = 0, \quad \eta = \delta_{\alpha\kappa} x_\alpha, \tag{4.15}$$

where κ is a fixed number.

Let

$$\mathscr{C}_{\alpha(\kappa)} := P_\alpha^{(\mathrm{var})}\Big|_{\{a_5=a_6=1, \text{ and } a_i=0, i\neq 5,6\}}. \tag{4.16}$$

We have dual conservation law

$$\mathscr{C}_{\alpha(\kappa)} = \frac{1}{E} x_\kappa \varphi_{,\beta\beta\alpha} + \delta_{\beta\kappa}\left(\frac{1+v}{E}\varphi_{,\alpha\beta} - \frac{v}{E}\varphi_{,\tau\tau}\delta_{\alpha\beta}\right), \tag{4.17}$$

where $\alpha, \kappa = 1, 2$.

(5) Gauss Theorem (divergence theorem).

Assume $a_7 = 1$ and $a_i = 0$, $i \neq 7$. Then

$$\xi_\alpha = 0; \quad \eta = 1. \tag{4.18}$$

Let

$$\mathscr{G}_\alpha := P_\alpha^{(\mathrm{var})}\Big|_{\{a_7=1\}}. \tag{4.19}$$

We recover the Gauss (divergence) theorem

$$\mathscr{G}_\alpha = \frac{1}{E}\varphi_{,\beta\beta\alpha}. \tag{4.20}$$

(6) Inversion.

When $v = 1/2$ for plane strain state, additional conservation laws are valid. Assume $a_8 = \delta_{1\kappa}$, $a_9 = \delta_{2\kappa}$ for a fixed κ, and $a_i = 0$, $i \neq 8, 9$. The infinitesimal generators have the forms

$$\xi_\alpha = 2x_\alpha x_\kappa - \delta_{\alpha\kappa} x_\lambda x_\lambda, \tag{4.21}$$

$$\eta = 2x_\kappa \varphi. \tag{4.22}$$

Subsequently,

$$\eta - \xi_\gamma \varphi_{,\gamma} = 2x_\kappa\varphi - (2x_\gamma x_\kappa - \delta_{\gamma\kappa} x_\lambda x_\lambda)\varphi_{,\gamma} \tag{4.23}$$

$$D_\beta(\eta - \xi_\gamma\varphi_{,\gamma}) = 2\delta_{\beta\kappa}\varphi - 2(\delta_{\beta\kappa} x_\gamma - \delta_{\gamma\kappa} x_\beta)\varphi_{,\gamma} - 2(2x_\gamma x_\kappa - \delta_{\gamma\kappa} x_\lambda x_\lambda)\varphi_{,\gamma\beta}. \tag{4.24}$$

Define

$$\mathscr{I}_{\alpha(\kappa)} := P_\alpha^{(\mathrm{var})}\Big|_{\{a_8\neq 0, a_9\neq 0\}}. \tag{4.25}$$

It then follows that

$$\mathscr{I}_{\alpha(\kappa)} = (2x_\alpha x_\kappa - \delta_{\alpha\kappa}x_\lambda x_\lambda)L_c - \frac{(1-v^2)}{E}\big(2x_\kappa\varphi - (2x_\kappa x_\gamma - \delta_{\gamma\kappa}x_\lambda x_\lambda)\varphi_{,\gamma}\big)\varphi_{,\beta\beta\alpha}$$
$$+ \frac{(1+v)}{E}\big\{2\delta_{\beta\kappa}\varphi - 2(\delta_{\beta\kappa}x_\gamma + \delta_{\gamma\kappa}x_\beta)\varphi_{,\gamma} - (2x_\gamma x_\kappa - \delta_{\gamma\kappa}x_\lambda x_\lambda)\varphi_{,\beta\gamma}\big\}\big(\varphi_{,\alpha\beta} - v\varphi_{,\tau\tau}\delta_{\alpha\beta}\big), \tag{4.26}$$

where α, $\kappa = 1, 2$.

4.2. Bessel–Hagen type conservation laws

The dual conservation laws in this category involve with a term, B_α, which can be introduced by a null Lagrangian. A few examples of the Bessel–Hagen type conservation laws are presented in the followings.

(1) Reciprocal formula of Betti–Rayleigh type.

Let

$$\xi_\alpha = 0, \quad \eta = \psi; \quad \text{and} \quad B_\alpha = C_{\alpha\beta\lambda\mu}\psi_{,\lambda\mu}\varphi_{,\beta}, \tag{4.27}$$

where ψ is a solution of biharmonic equation, i.e. $\psi_{,\mu\mu\lambda\lambda} = 0$. We then have the following conservation law,

$$\mathscr{H}^{(\mathrm{I}a)}_\alpha = -\frac{1}{E}\psi\varphi_{,\beta\beta\alpha} + \psi_{,\beta}\left(\frac{1+v}{E}\varphi_{,\alpha\beta} - \frac{v}{E}\varphi_{,\tau\tau}\delta_{\alpha\beta}\right) - \varphi_{,\beta}\left(\frac{1+v}{E}\psi_{,\alpha\beta} - \frac{v}{E}\psi_{,\tau\tau}\delta_{\alpha\beta}\right), \tag{4.28}$$

where $C_{\alpha\beta\lambda\mu}$ is the elastic tensor.

(2) Reciprocal formula of Green type.

Suppose that both φ and ψ are the solutions of biharmonic equation. Since $\nabla^2\nabla^2$ is self-adjoint, a simple integration by part yields,

$$\psi\varphi_{,\alpha\alpha\beta\beta} - \varphi\psi_{,\alpha\alpha\beta\beta} = \big[\psi\varphi_{,\alpha\alpha\beta} - \psi_{,\beta}\varphi_{,\alpha\alpha} - \varphi\psi_{,\alpha\alpha\beta} + \varphi_{,\beta}\psi_{,\alpha\alpha}\big]_{,\beta} = 0. \tag{4.29}$$

It yields the conservation law,

$$\mathscr{H}^{(\mathrm{II}a)}_\beta = \psi\varphi_{,\alpha\alpha\beta} - \psi_{,\beta}\varphi_{,\alpha\alpha} - \varphi\psi_{,\alpha\alpha\beta} + \varphi_{,\beta}\psi_{,\alpha\alpha}. \tag{4.30}$$

In particular, let $\psi = b_{\mathrm{I}\gamma}\varphi_{,\gamma}$. We have the higher order conservation law:

$$\mathscr{H}^{(\mathrm{II}b)}_\beta = b_{\mathrm{I}\gamma}\big(\varphi_{,\gamma}\varphi_{,\alpha\alpha\beta} - \varphi_{,\gamma\beta}\varphi_{,\alpha\alpha} - \varphi\varphi_{,\gamma\alpha\alpha\beta} + \varphi_{,\beta}\varphi_{,\gamma\alpha\alpha}\big). \tag{4.31}$$

Let $\psi = b_{\mathrm{II}\xi}x_\xi\varphi_{,\eta\eta}$. It is readily shown that $\psi_{,\alpha\alpha\beta\beta} = 0$. Therefore, we have

$$\mathscr{H}^{(\mathrm{II}c)}_\beta = b_{\mathrm{II}\xi}x_\xi\varphi_{,\eta\eta}\varphi_{,\alpha\alpha\beta} - (b_{\mathrm{II}\beta}\varphi_{,\zeta\zeta} + b_{\mathrm{II}\xi}x_\xi\varphi_{,\xi\xi\beta})\varphi_{,\alpha\alpha} - 2b_{\mathrm{II}\alpha}\varphi_{,\xi\xi\alpha\beta}\varphi + 2b_{\mathrm{II}\alpha}\varphi_{,\xi\xi\alpha}\varphi_{,\beta}. \tag{4.32}$$

More higher order conservation laws can be obtained by letting,

$$\psi = b_{\mathrm{II}\gamma,\dots,\zeta\eta}\varphi_{,\gamma,\dots,\zeta\eta}. \tag{4.33}$$

Some conservation laws in this class may be equivalent to the first class.

(3) Higher order conservation law (III).

Let

$$\xi_\alpha = 0, \quad \eta = \varphi_{,\gamma\gamma}. \tag{4.34}$$

Hence $\eta_\beta = \varphi_{,\gamma\gamma\beta}$. Choose

$$B_\alpha = C_{\alpha\beta\lambda\mu}\varphi_{,\beta}\varphi_{,\gamma\gamma\lambda\mu}. \tag{4.35}$$

It then follows

$$\mathscr{H}^{(\mathrm{III})}_\alpha = -\frac{1}{E}\varphi_{,\gamma\gamma}\varphi_{,\beta\beta\alpha} + C_{\alpha\beta\lambda\mu}\varphi_{,\lambda\mu}\varphi_{,\gamma\gamma\beta} - C_{\alpha\beta\lambda\mu}\varphi_{,\gamma\gamma\lambda\mu}\varphi_{,\beta}. \tag{4.36}$$

(4) Higher order conservation law (IV).

Let

$$\xi_\alpha = 0, \quad \eta = \delta_{\lambda\kappa}\varphi_{,\lambda\mu\mu} = \varphi_{,\kappa\mu\mu}, \tag{4.37}$$

for a fixed number κ: 1 or 2. Therefore $\eta_{,\beta} = \varphi_{,\mu\mu\beta\kappa}$. The corresponding null divergence is

$$B_{\alpha(\kappa)} = C_{\alpha\beta\lambda\mu}\varphi_{,\lambda\mu\gamma\gamma\kappa}\varphi_{,\beta}. \tag{4.38}$$

We have the following conservation law

$$\mathscr{H}^{(\mathrm{IV})}_{\alpha(\kappa)} = -\frac{1}{E}\varphi_{,\lambda\lambda\kappa}\varphi_{,\beta\beta\alpha} + C_{\alpha\beta\lambda\mu}\varphi_{,\lambda\mu}\varphi_{,\mu\mu\beta\kappa} - C_{\alpha\beta\lambda\mu}\varphi_{,\lambda\mu\gamma\gamma\kappa}\varphi_{,\beta}. \tag{4.39}$$

(5) Higher order conservation law (V).

Let

$$\xi_\alpha = 0; \quad \eta = \epsilon_{\kappa\gamma}x_\gamma\varphi_{,\kappa\mu\mu}. \tag{4.40}$$

Choosing

$$B_\alpha = \frac{1+v}{E}\left(2[\epsilon_{\kappa\beta}\varphi_{,\mu\mu\kappa\alpha}\varphi_{,\beta}] + [\epsilon_{\kappa\gamma}x_\gamma\varphi_{,\lambda\mu\mu\alpha\beta}\varphi_{,\beta}]\right) \tag{4.41}$$

we have higher order conservation law

$$\mathscr{H}^{(\mathrm{V})}_{\alpha} = -\frac{1}{E}\epsilon_{\kappa\gamma}x_{\gamma}\varphi_{,\kappa\mu\mu}\varphi_{,\beta\beta\alpha} + C_{\alpha\beta\lambda\mu}\varphi_{,\lambda\mu}\big(\epsilon_{\kappa\beta}\varphi_{,\kappa\mu\mu} + \epsilon_{\kappa\gamma}x_{\gamma}\varphi_{,\lambda\lambda\kappa\beta}\big) - \frac{1+\nu}{E}\big(2[\epsilon_{\kappa\beta}\varphi_{,\mu\mu\kappa\alpha}\varphi_{,\beta}] + [\epsilon_{\kappa\gamma}x_{\gamma}\varphi_{,\kappa\mu\mu\alpha\beta}\varphi_{,\beta}]\big). \tag{4.42}$$

5. Eshelby tensor and dual Eshelby tensors

The dual conservation laws found in this paper are genuine and non-trivial conservation laws. Most of them have not been known before. They are different from the conservation laws derived by Knowles and Sternberg [15] and Olver [25,26]. They represent the intrinsic symmetry properties of compatibility conditions, whereas the previous conservation laws represent the symmetry properties of Navier equations.

To compare the differences between the dual conservations laws derived here with previous conservation laws in elasticity, we consider a quantity that has most significant meaning in physics, i.e. Eshelby's energy momentum tensor. There are two dual-Eshelby tensors: one derived by Bui [3] and the one found in this paper (Eq. (4.13)).

The Eshelby's energy momentum tensor is defined as

$$\mathscr{E}_{\alpha\beta} := W\delta_{\alpha\beta} - u_{\gamma,\alpha}\sigma_{\gamma\beta} \tag{5.1}$$

and Bui's dual energy momentum tensor is defined as

$$\mathscr{B}_{\alpha\beta} := L_c\delta_{\alpha\beta} - u_{\gamma}\sigma_{\gamma\beta,\alpha}. \tag{5.2}$$

Recalling the definition (2.7) and the identity (2.9), one may be able to show that

$$\varphi_{,\beta\beta\alpha} = \epsilon_{\beta\lambda}\epsilon_{\beta\mu}\sigma_{\lambda\mu,\alpha} = \delta_{\lambda\mu}\sigma_{\lambda\mu} = \sigma_{\mu\mu,\alpha}; \tag{5.3}$$

$$\varphi_{,\gamma\gamma} = \epsilon_{\gamma\lambda}\epsilon_{\gamma\mu}\sigma_{\lambda\mu} = \delta_{\lambda\mu}\sigma_{\lambda\mu} = \sigma_{\gamma\gamma}; \tag{5.4}$$

$$\frac{1+\nu}{E}\varphi_{,\alpha\beta} - \frac{\nu}{E}\varphi_{,\gamma\gamma}\delta_{\alpha\beta} = \epsilon_{\alpha\lambda}\epsilon_{\beta\mu}\varepsilon_{\lambda\mu}. \tag{5.5}$$

According to Timoshenko and Goodier [32], the displacement field may be expressed in terms of Airy stress function as,

$$u_{\alpha} = \frac{1}{2G}\left(-\varphi_{,\alpha} + \frac{4}{1+\nu}p_{\alpha}\right), \quad G = \frac{E}{2(1+\nu)}, \tag{5.6}$$

where p_{α}, ($\alpha = 1, 2$) is a pair of conjugate harmonic functions, and they are related to the Airy stress function by definition or identity, $p_{\alpha,\alpha} = \frac{1}{2}\varphi_{,\alpha\alpha}$.

Eshelby's energy momentum tensor may be written in terms of Airy stress function

$$\mathscr{E}_{\alpha\beta} = W\delta_{\alpha\beta} - \epsilon_{\alpha\zeta}\epsilon_{\beta\eta}\left(\frac{1+\nu}{E}\varphi_{,\zeta\gamma}\varphi_{,\eta\gamma} - \frac{\nu}{E}\varphi_{,\zeta\eta}\varphi_{,\gamma\gamma}\right) - \frac{2}{E}(p_{\gamma,\alpha} - p_{\alpha,\gamma})\epsilon_{\gamma\zeta}\epsilon_{\beta\eta}\varphi_{,\zeta\eta}. \tag{5.7}$$

On the other hand, Bui's dual energy–momentum tensor may be expressed in terms of Airy stress function as

$$\mathscr{B}_{\alpha\beta} = L_c\delta_{\alpha\beta} - \frac{1+\nu}{E}\left(-\varphi_{,\gamma} + \frac{4}{1+\nu}p_\gamma \right)\epsilon_{\gamma\zeta}\epsilon_{\beta\eta}\varphi_{,\zeta\eta\alpha}. \tag{5.8}$$

Note that both Eshelby's energy momentum tensor and Bui's dual energy momentum tensor can not be completely determined by the Airy stress function, unless additional conditions specified. Assume that $p_{\gamma,\alpha} = p_{\alpha,\gamma}$ and $2p_\gamma = \varphi_{,\gamma}$. [2] We may have

$$\mathscr{E}_{\alpha\beta} = W\delta_{\alpha\beta} - \epsilon_{\alpha\zeta}\epsilon_{\beta\eta}\left(\frac{1+\nu}{E}\varphi_{,\zeta\gamma}\varphi_{,\eta\gamma} - \frac{\nu}{E}\varphi_{,\zeta\eta}\varphi_{,\gamma\gamma} \right) \tag{5.9}$$

$$\mathscr{B}_{\alpha\beta} = L_c\delta_{\alpha\beta} - \frac{1-\nu}{E}\epsilon_{\gamma\zeta}\epsilon_{\beta\eta}\varphi_{,\gamma}\varphi_{,\zeta\eta\alpha}. \tag{5.10}$$

Compare with dual Eshelby tensor obtained in this paper

$$\mathscr{L}_{\alpha\beta} := \mathscr{T}_{\beta(\alpha)} = L_c\delta_{\alpha\beta} + \frac{1}{E}\varphi_{,\alpha}\varphi_{,\beta\beta\alpha} - \left(\frac{1+\nu}{E}\varphi_{,\beta\kappa}\varphi_{,\alpha\kappa} - \frac{\nu}{E}\varphi_{,\alpha\beta}\varphi_{,\gamma\gamma} \right). \tag{5.11}$$

Note that for linear elastic materials $W = L_c$.

It is obvious that the differences among them are distinct and non-trivial. According to Eqs. (5.9) and (5.10), it seems that the dual Eshelby tensor derived in this paper has features in both original Eshelby's energy momentum tensor and Bui's dual-energy momentum tensor. An in-depth study is needed to interpret physical meanings of their differences.

6. Closure

Most dual conservation laws derived in this paper are new except the conservation laws corresponding to the Gauss (divergence) theorem and the Betti–Rayleigh reciprocal formula.

To indulge oneself, one may venture to speculate that the dual conservation laws obtained here may provide foundation to a new class of "material tensors" that are characterized by deformation compatibility requirements, such as constraints for dislocations or disinclinations in solids. These material momentum tensors may be expressed in stress function space. In short, this study suggests that there may exist two types of material momentum tensors, Eshelbian type in deformation space and dual Eshelbian type in stress function space.

Furthermore, using stress function formalism to derive conservation laws has practical interests. It is well known that the Airy stress function can be expressed by two analytical functions, a fact guaranteed by Goursat's theorem (see [23]). In fact, the stress function related complex variable formulation has been a primary method used to solve many engineering problems such as

[2] They are not true in general.

crack problems, and the success of fracture mechanics owes a great deal to it. The dual conservation laws obtained in this paper may allow us to make an easy link between invariant path-integrals and Muskhelishvili's complex potentials.

The same procedure can be readily extended into three-dimensional (3D) elasticity. Dual conservation laws can be derived based on general Maxwell–Morera stress function formalism. The 3D extension of this paper will be discussed in a separate paper.

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