



## Micromechanics modeling of plastic yielding in a solid containing mode III cohesive cracks

**Shaofan Li**

Department of Civil and Environmental Engineering,  
University of California, Berkeley, CA94720, USA  
*e-mail: li@ce.berkeley.edu*

**Elise F. Morgan**

Department of Mechanical Engineering,  
University of California, Berkeley, CA94720, USA  
*e-mail: emorgan@me.berkeley.edu*

**Abstract.** A new micromechanics damage model is proposed by averaging distributed microcracks with cohesive zones in a two dimensional representative volume element. The cohesive microcracks are mode-III Dugdale-Bilby-Cottrell-Swinden (Dugdale-BCS) crack. The damage model may be used to construct plasticity potentials that take into account the presence of such microcracks.

**Key words:** Cohesive crack, Fracture, Micromechanics, Plasticity

**1. Introduction.** Since Barenblatt (1959) and Dugdale's pioneer contributions (1960), cohesive crack models have been studied extensively, because of their micro-mechanics interpretation of atomistic force interactions. The cohesive micro-crack model mimics realistic interactions among various bond forces at micro-scale. Therefore a statistical average of such interactions may capture overall damage effect in a material due to bond breaking. In this note, micromechanics procedure is applied to evaluate the damage effect of distributed Dugdale-BCS (Mode III) cracks on overall plastic yielding.

### 2. Dugdale-BCS crack solution (mode III).

The mode III (anti-plane) Dugdale-BCS crack problem (Dugdale (1960) and Bilby et al (1963)) has become a classical example of cohesive fracture. Consider a single Dugdale-BCS crack as shown in Fig. 1. The displacement along  $x_1$ -axis ( $x_2 = 0$ ) (e.g. Mura (1987)) is

$$w(x_1, 0) = \frac{\sigma_c}{\pi\mu} \left\{ x_1 \ln \left| \frac{x_1\sqrt{b^2 - a^2} - a\sqrt{b^2 - x_1^2}}{x_1\sqrt{b^2 - a^2} + a\sqrt{b^2 - x_1^2}} \right| \right\}$$

$$-a \ln \left| \frac{\sqrt{b^2 - a^2} - \sqrt{b^2 - x_1^2}}{\sqrt{b^2 - a^2} + \sqrt{b^2 - x_1^2}} \right| \Bigg\} , \quad (1)$$

where  $b-a$  is the length of the cohesive zone ( $a/b = \cos(\pi\Sigma_\infty/2\sigma_c)$ ) and  $\sigma_c$  is the cohesive strength.

At the crack tip  $x_1 = a$ , the crack tip opening (slip) displacement is

$$\delta_t = w(a, 0^+) - w(a, 0^-) = \frac{4\sigma_c a}{\pi\mu} \ln \frac{b}{a} . \quad (2)$$

Under small-scale yielding condition ( $a/b \approx 1$ ),

$$w(a, 0) = \frac{2\sigma_c a}{\pi\mu} \ln \frac{b}{a} \approx \frac{2\sigma_c a}{\pi\mu} \left( \frac{b}{a} - 1 \right) = \frac{2\sigma_c}{\pi\mu} (b - a) . \quad (3)$$

Inside the cohesive zone,

$$\sigma_{13} = \mu \frac{\partial w}{\partial x_1} \approx \mu \frac{w(b, 0) - w(a, 0)}{b - a} = -\frac{2\sigma_c}{\pi} , \quad (4)$$

$$\sigma_{23} = \sigma_c . \quad (5)$$

We assume that micro plastic yielding in an RVE obeys the Huber-von Mises criterion. The cohesive strength  $\sigma_c$  can then be related to the true yield stress by

$$\frac{\sigma_Y}{\sigma_c} = \frac{\sqrt{3(\pi^2 + 4)}}{\pi} . \quad (6)$$

The traction-free surface crack opening volume is:

$$\begin{aligned} V(a) &= \int_{-a}^a [w(x_1, 0^+) - w(x_1, 0^-)] dx_1 \\ &= \frac{2\sigma_c a^2}{\pi\mu} \left\{ \left(1 - \frac{\Sigma_\infty}{\sigma_c}\right) \tan\left(\frac{\pi\Sigma_\infty}{2\sigma_c}\right) - \frac{4}{\pi} \ln \left[ \cos\left(\frac{\pi\Sigma_\infty}{2\sigma_c}\right) \right] \right\} . \end{aligned} \quad (7)$$

To include non-traction-free surface slip, one can calculate the total crack surface slip,

$$V(b) = \int_{-b}^b [w(x_1, 0^+) - w(x_1, 0^-)] dx = \frac{2\sigma_c a^2}{\mu} \tan\left(\frac{\pi\Sigma_\infty}{2\sigma_c}\right) , \quad (8)$$

which may serve as a conservative estimate on crack surface separation volume.

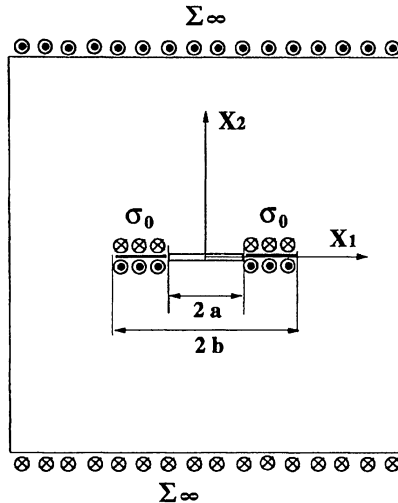


Figure 1: An anti-plane cohesive crack.

As shown by Rice [1968], the J-integral related energy release rate is

$$-\frac{\partial \Delta \Pi}{\partial \ell} = J = \frac{4\sigma_c^2 a}{\pi \mu} \ln \left[ \sec \left( \frac{\pi \Sigma_\infty}{2\sigma_c} \right) \right], \quad (9)$$

which may be interpreted as energy release rate due to traction-free surface separation. Note that  $\ell = 2a$  is the length of traction-free portion of the crack, and  $\Delta \Pi$  is the change of the potential energy due to traction-free surface separation. Denote the energy release as  $\mathcal{R}_1 = -\Delta \Pi$  and assume that during crack growth the ratio,  $\Sigma_\infty/\sigma_c$ , remains constant. Integrating (9), one may find the energy release of a single Dugdale-BCS crack

$$\mathcal{R}_1 = \frac{4\sigma_c^2 a^2}{\pi \mu} \ln \left[ \sec \left( \frac{\pi \Sigma_\infty}{2\sigma_c} \right) \right]. \quad (10)$$

The energy release expression given in Eq. (10) does not include plastic dissipation, nor is it the total crack separation energy release (see Kfoury (1979)). It is smaller than the actual energy loss in the damage process, therefore it is a lower bound estimate, or a conservative estimate.

To seek an upper bound estimate, one may overestimate energy release by assuming that all the energy loss is consumed in surface separation, i.e.

$$\mathcal{R}_2 = \int_{-a}^a \Sigma_\infty [w](x_1) dx_1 + 2 \int_a^b (\Sigma_\infty - \sigma_0) [w](x_1) dx_1, \quad (11)$$

where  $[w](x) = w(x_1, 0^+) - w(x_1, 0^-)$ . It is straightforward to show

$$\mathcal{R}_2 = \Sigma_\infty V(b) - \sigma_c(V(b) - V(a)) = \frac{8\sigma_c^2 a^2}{\pi\mu^*} \ln \left[ \sec \left( \frac{\pi\Sigma_\infty}{2\sigma_c} \right) \right]. \quad (12)$$

Note that  $\mathcal{R}_2 = 2\mathcal{R}_1$ . Combining (10) and (12), we write

$$\mathcal{R}_\omega = \frac{4\omega\sigma_c^2 a^2}{\pi\mu^*} \ln \left[ \sec \left( \frac{\pi\Sigma_\infty}{2\sigma_c} \right) \right], \quad \omega = 1, 2 \quad (13)$$

**3. Effective properties of a solid with cohesive micro-cracks (mode III).** There are two ways to find the effective elastic compliances for a RVE containing distributed cohesive cracks: (1) use Hill's additional strain formula for traction-free defects (Hill [1963] and Kachanov [1992]), (2) use energy methods.

The additional strain caused by a crack  $\Omega$  may be estimated by Hill's formula,

$$\epsilon^{(add)} = \frac{1}{2V} \int_{\partial\Omega} (\mathbf{n} \otimes [\mathbf{u}] + [\mathbf{u}] \otimes \mathbf{n}) dS, \quad (14)$$

where the superscript *add* stands for *additional* strain, and  $\partial\Omega$  is crack surface boundary.

For cohesive cracks, there are two crack surface separations: traction-free surface separation and total crack surface separation. Hence, there are two different estimates:

$$\begin{aligned} \epsilon_{23}^{(add)} &= \frac{1}{2V} \int_{\partial\Omega} n_2 [u_3] dS = \frac{1}{2V} \begin{cases} V(a) \\ V(b) \end{cases} \\ &= \frac{\sigma_c}{\pi\mu^*} \left( \frac{\pi a^2}{V} \right) \begin{cases} \left( 1 - \frac{\Sigma_\infty}{\sigma_c} \right) \tan \left( \frac{\pi\Sigma_\infty}{2\sigma_c} \right) - \frac{4}{\pi} \ln \left[ \cos \left( \frac{\pi\Sigma_\infty}{2\sigma_c} \right) \right] \\ \tan \left( \frac{\pi\Sigma_\infty}{2\sigma_c} \right) \end{cases} \end{aligned} \quad (15)$$

From  $\mathcal{E}_{23} = \epsilon_{23}^{(0)} + \epsilon_{23}^{(add)}$ , one may find that

$$\frac{\Sigma_\infty}{2\bar{\mu}} = \frac{\Sigma_\infty}{2\mu} + \frac{\Sigma_\infty}{\pi\mu^*} \left( \frac{\pi a^2}{V} \right) \left( \frac{\sigma_0}{\Sigma_\infty} \right) \begin{cases} \left( 1 - \frac{\Sigma_\infty}{\sigma_c} \right) \tan \left( \frac{\pi\Sigma_\infty}{2\sigma_c} \right) - \frac{4}{\pi} \ln \cos \left( \frac{\pi\Sigma_\infty}{2\sigma_c} \right) \\ \tan \left( \frac{\pi\Sigma_\infty}{2\sigma_c} \right) \end{cases} \quad (16)$$

Suppose that there are  $N$  parallel mode III cohesive cracks randomly distributed inside a RVE, and each of them having length  $2a_\ell$ ,  $\ell = 1, 2, \dots, N$ . Define the crack density parameter,

$$f := \sum_{\ell=1}^N \frac{\pi a_\ell^2}{V} \quad (17)$$

and use the self-consistent scheme (e.g. Budiansky and O'Connell [1976]). One can then find the effective shear modulus as,

$$\frac{\bar{\mu}}{\mu} = 1 - \frac{2f}{\pi} B\left(\frac{\Sigma_\infty}{\sigma_c}\right) \quad (18)$$

where

$$B\left(\frac{\Sigma_\infty}{\sigma_c}\right) := \left(\frac{\sigma_c}{\Sigma_\infty}\right) \begin{cases} \left(1 - \frac{\Sigma_\infty}{\sigma_c}\right) \tan\left(\frac{\pi \Sigma_\infty}{2\sigma_c}\right) - \frac{4}{\pi} \ln \cos\left(\frac{\pi \Sigma_\infty}{2\sigma_c}\right), & ; \\ \tan\left(\frac{\pi \Sigma_\infty}{2\sigma_c}\right), \end{cases} \quad (19)$$

On the other hand, if the crack distribution is dilute, one may find

$$\frac{\mu}{\bar{\mu}} = 1 + \frac{2f}{\pi} B\left(\frac{\Sigma_\infty}{\sigma_c}\right) \quad (20)$$

Eqs. (18) and (20) show the dependence of the effective shear modulus,  $\bar{\mu}$ , on the ratio  $\Sigma_\infty/\sigma_c$ .

Now we use energy method to estimate effective material properties. Assume that the complementary potential energy density of a virgin material is  $W^c = \frac{1}{2\mu} \Sigma_\infty^2$ . The overall complementary energy density can be obtained by considering strain energy balance

$$\bar{W} = \left( \langle \sigma_{ij} \epsilon_{ij} \rangle - W^c \right) - \frac{\mathcal{R}_\omega}{V} \Rightarrow \bar{W}^c = W^c + \frac{\mathcal{R}_\omega}{V}, \quad (21)$$

where  $\bar{W}$  is the overall strain energy density. Therefore,

$$\mathcal{E}_{23} = \frac{\partial \bar{W}^c}{\partial \Sigma_{23}} = \epsilon_{23}^{(0)} + \frac{2\omega \sigma_c f}{\pi \mu} \tan\left(\frac{\pi \Sigma_\infty}{2\sigma_c}\right) \quad (22)$$

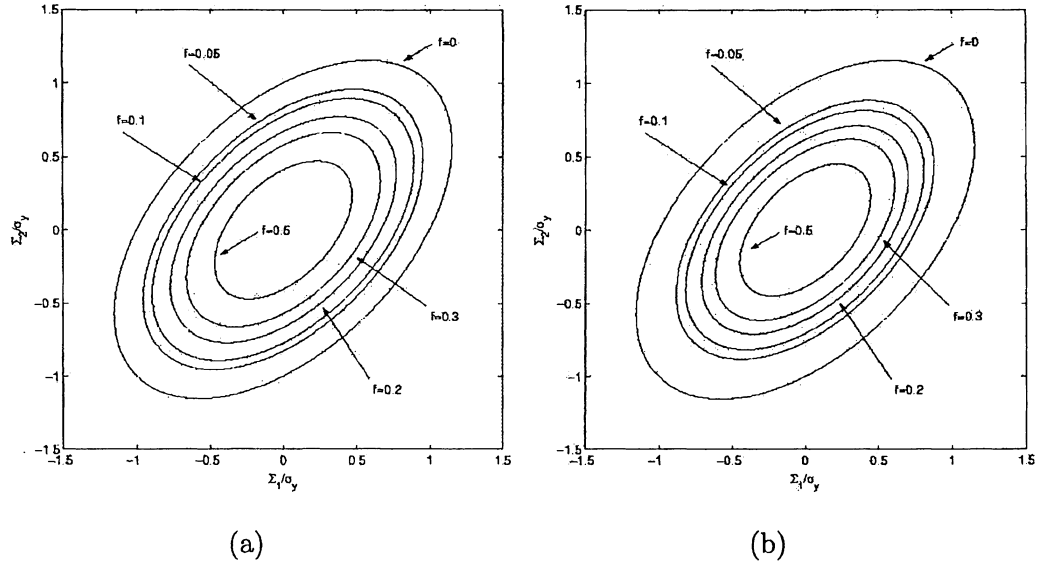


Figure 2: One dimensional damage models (a) Model 1a (based on Hill's formula ); (b) Model 2a (based on energy method ).

Consider the self-consistent scheme. One may find the ratio between the effective shear modulus and the initial shear modulus

$$\frac{\bar{\mu}}{\mu} = 1 - \omega f \left( \frac{2\sigma_c}{\pi\Sigma_\infty} \right) \tan\left(\frac{\pi\Sigma_\infty}{2\sigma_c}\right), \quad \omega = 1, 2 \quad (23)$$

It is interesting to note that when  $\omega = 1$  (23) is exactly the same as (18) and (19)<sub>2</sub>, which are obtained via Hill's formula. Whereas when  $\omega = 2$ , the energy method gives different estimates from the estimates obtained by Hill's formula. If the micro-crack distribution is dilute, it can be shown that

$$\frac{\mu}{\bar{\mu}} = 1 + \omega f \left( \frac{2\sigma_c}{\pi\Sigma_\infty} \right) \tan\left(\frac{\pi\Sigma_\infty}{2\sigma_c}\right), \quad \omega = 1, 2 . \quad (24)$$

It should be noted that after averaging the material becomes transversely isotropic and the effective shear modulus provided above is actually  $\bar{\mu} = \bar{\mu}_{13} = \bar{\mu}_{23}$ .

#### 4. Damage effect of cohesive crack on macro plastic yielding.

To derive new plastic yielding potentials, we assume that the transversely

isotropic RVE may be viewed as an isotropic RVE in approximation, and we further postulate that *the maximum macro elastic octahedral strain in a given material ensemble is finite*. Therefore, the macroscopic yielding of an RVE begins when macro elastic octahedral strain reaches a threshold, i.e.

$$\mathcal{E}_{oct} := \frac{2\sqrt{2}}{\sqrt{3}} \sqrt{I_2} = \epsilon_{cr} \quad (25)$$

where  $I_2 = \frac{1}{2} \mathcal{E}'_{ij} \mathcal{E}'_{ij}$  and  $\mathcal{E}'_{ij}$  are calculated based on elastic loading, i.e.  $\mathcal{E}'_{ij} = \frac{1}{2\bar{\mu}} \Sigma'_{ij}$ . Note that this is slightly different from the micro-yielding criterion, which is assumed as the Huber-von Mises criterion based on the maximum distortion-energy theory.

The maximum macro elastic octahedral strain criterion can then be expressed as

$$\mathcal{E}_{oct} = \frac{1}{3\sqrt{2}\bar{\mu}} \Sigma_{eq} \leq \frac{1}{3\sqrt{2}\mu} \sigma_Y \Rightarrow \frac{\Sigma_{eq}}{\sigma_Y} = \frac{\bar{\mu}}{\mu} \quad (26)$$

Considering (6) and  $\Sigma_{eq} = \sqrt{3}\Sigma_{\infty}$ , we have

$$\frac{\Sigma_{\infty}}{\sigma_c} = \sqrt{1 + \frac{4}{\pi^2} \left( \frac{\Sigma_{eq}}{\sigma_Y} \right)} \quad (27)$$

By substituting (18) and (19<sub>1,2</sub>) into (26), the following plastic potentials may be derived

Damage Model 1a (Hill's formula & self – consistent) :

$$\begin{aligned} & \frac{\Sigma_{eq}^2}{\sigma_Y^2} + \frac{2f}{\sqrt{\pi^2 + 4}} \left\{ \left( 1 - \sqrt{1 + \frac{4}{\pi^2} \left( \frac{\Sigma_{eq}}{\sigma_Y} \right)} \right) \tan\left( \frac{\sqrt{\pi^2 + 4}\Sigma_{eq}}{2\sigma_Y} \right) \right. \\ & \left. - \frac{4}{\pi} \ln \left[ \cos\left( \frac{\sqrt{\pi^2 + 4}\Sigma_{eq}}{2\sigma_Y} \right) \right] \right\} - \frac{\Sigma_{eq}}{\sigma_Y} = 0 \end{aligned} \quad (28)$$

Damage Model 1b (Hill's formula & dilute crack distribution) :

$$\begin{aligned} & \frac{\Sigma_{eq}^2}{\sigma_Y^2} + \frac{2f\Sigma_{eq}}{\sigma_Y\sqrt{\pi^2 + 4}} \left\{ \left( 1 - \sqrt{1 + \frac{4}{\pi^2} \left( \frac{\Sigma_{eq}}{\sigma_Y} \right)} \right) \tan\left( \frac{\sqrt{\pi^2 + 4}\Sigma_{eq}}{2\sigma_Y} \right) \right. \\ & \left. - \frac{4}{\pi} \ln \left[ \cos\left( \frac{\sqrt{\pi^2 + 4}\Sigma_{eq}}{2\sigma_Y} \right) \right] \right\} - \frac{\Sigma_{eq}}{\sigma_Y} = 0 \end{aligned} \quad (29)$$

Damage Model 2a (Energy method & self – consistent) :

$$\frac{\Sigma_{eq}^2}{\sigma_Y^2} + \frac{2\omega f}{\sqrt{\pi^2 + 4}} \tan\left(\frac{\sqrt{\pi^2 + 4}\Sigma_{eq}}{2\sigma_Y}\right) - \frac{\Sigma_{eq}}{\sigma_Y} = 0, \quad \omega = 1, 2$$

Damage Model 2b (Energy method & dilute crack distribution) :

$$\frac{\Sigma_{eq}^2}{\sigma_Y^2} + \frac{2\omega f \Sigma_{eq}}{\sigma_Y \sqrt{\pi^2 + 4}} \tan\left(\frac{\sqrt{\pi^2 + 4}\Sigma_{eq}}{2\sigma_Y}\right) - \frac{\Sigma_{eq}}{\sigma_Y} = 0, \quad \omega = 1, 2$$

In Fig.2, effective yield surfaces are plotted in a two-dimensional stress space, which are a set of coaxial ellipses with different crack volume fraction values.

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