On Global Energy Release Rate of a Permeable Crack in a Piezoelectric Ceramic

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A permeable crack model is proposed to analyze crack growth in a piezoelectric ceramic. In this model, a permeable crack is modeled as a vanishing thin, finite dimension, rectangular slit with dielectric medium inside. A first-order approximation solution is derived in terms of the slit height, h_0. The main contribution of this paper is that the newly proposed permeable crack model reveals that there exists a realistic leaky mode for electrical field, which allows applied electric field passing through the dielectric medium inside a crack. By taking into account the leaky mode effect, a correct estimation of electrical and mechanical fields in front of a crack tip in a piezoelectric ceramic is obtained. To demonstrate this new finding, a closed-form solution is obtained for a mode III permeable crack under both mechanical as well electrical loads. Both local and global energy release rates are calculated based on the permeable crack solution obtained. It is found that the global energy release rate derived for a permeable crack is in a broad agreement with some known experimental observations. It may be served as a fracture criterion for piezoelectric materials. This contribution reconciles the outstanding discrepancy between experimental observation and theoretical analysis on crack growth problem in piezoelectric materials. [DOI: 10.1115/1.1544539]

1 Introduction

Fracture mechanics of piezoelectric solids has been an active research area since early 1990s due to the widespread use of smart materials and smart structures. Many research works have been published in the past decade, e.g., Pak [1,2], Li et al. [3], Sosa [4,5], Suo et al. [6,7], Dunn [8], Dascalu and Maugin [9,10], Park and Sun [11,12], Gao and Barnett [13], and Gao et al. [14], Lynch et al. [15,16], Zhang and Hack [17], Fulton and Gao [18], Ru [19,20], Yang and Zhu [21–23], Zhang et al. [24,25], McMeeking [26,27], Yang [21,22] among others. A recent article by Zhang et al. [28] provides an excellent review.

A major challenge in fracture mechanics of piezoelectric materials has been how to resolve an outstanding discrepancy between experimental observation and theoretic analysis. In a landmark experimental work by Park and Sun [11], it was found that the experimental observation contradicts with some basic aspects of fracture mechanics theory of linear piezoelectric materials. For instance, the experimental results obtained by Park and Sun [11] show that there is a decrease in the critical stress of a cracked piezoelectric body if the electric field is applied along the direction of poling axis, and there is an increase in critical stress if the electric field is applied to the opposite direction, whereas according to linear fracture mechanics theory, the applied electric field does not induce any nonzero stress intensity factor (e.g., Pak [1,2] and Suo et al. [6]), and it always predicts a negative definite energy release rate regardless the directions of the applied electric fields, which implies that the applied electric field always retards crack growth.

Using micromechanics concepts related to domain switching, Gao and his co-workers [13,14,18] argued that crack growth in a piezoelectric solid is a multiscale phenomenon, and the local energy release rate may be a critical factor in fracture process. A local energy release rate criterion was subsequently proposed to measure the fracture toughness of piezoelectric materials. The local energy release rate criterion is based on the so-called saturation-strip model, or equivalently an electric dipole distribution model, which is basically a domain switch strip-zone model that is taking into account the nonlinearity induced by the overall effect of domain switching. The saturation-strip model is the direct analogous of Dugdale crack in a cohesive elastic medium of classical fracture mechanics.

The local energy release rate criterion was an immediate success, because it provides a plausible explanation on Park-Sun's empirical formula of energy release rate, [11,12]. However, the dissipative nature of saturation-strip model seems to be a nuisance, e.g., [27].

In this work, a permeable crack model is carefully crafted to render a tractable solution for mode III crack, while retaining all the main features of a permeable crack. By doing so, it provides an opportunity to systematically reexamine the permeable crack solution of a piezoelectric ceramic.

2 Formulation of the Problem

Consider a crack with finite dimension in the middle of a transversely isotropic piezoelectric solid under the antiplane mechanical load and the in-plane electrical load. Let $x_1=X$ and $x_2=Y$. The relevant field variables are

$$u_1=u_2\equiv 0, \quad u_3=:w(X,Y);$$

$$E_3\equiv 0, \quad \mathbf{E}=-\frac{\partial\phi}{\partial X}\mathbf{e}_1-\frac{\partial\phi}{\partial Y}\mathbf{e}_2.$$

For the symmetry class of 6 mm piezoelectric crystal, or general piezoelectric composite possessing the same symmetry, the relevant constitutive equations are as follows (Auld [29]):

$$\sigma_{XZ}=c_{44}^E\frac{\partial w}{\partial X}+e_{15}\frac{\partial\phi}{\partial X} \tag{1}$$

$$\sigma_{YZ}=c_{44}^E\frac{\partial w}{\partial Y}+e_{15}\frac{\partial\phi}{\partial Y} \tag{2}$$

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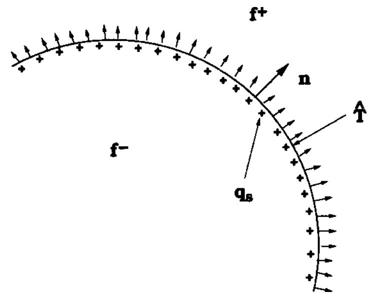

Fig. 1 Convention for boundary conditions

$$D_X = e_{15}\frac{\partial w}{\partial X} - \epsilon_{11}^s \frac{\partial \phi}{\partial X} \tag{3}$$

$$D_Y = e_{15}\frac{\partial w}{\partial Y} - \epsilon_{11}^s \frac{\partial \phi}{\partial Y}. \tag{4}$$

Subsequently, the Euler and Maxwell equations take the form

$$c_{44}^E \nabla^2 w + e_{15}\nabla^2 \phi = 0 \tag{5}$$

$$e_{15}\nabla^2 w - \epsilon_{11}^s \nabla^2 \phi = 0 \tag{6}$$

where

$$\nabla^2 := \frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2}.$$

Since the determinant

$$\Delta_i := -\begin{vmatrix} c_{44}^E & e_{15} \\ e_{15} & -\epsilon_{11}^s \end{vmatrix} \neq 0 \tag{7}$$

one can decouple the system of governing equations

$$\begin{cases} \nabla^2 w = 0, \quad \forall (X,Y) \in \mathbb{R}^2/\Omega_h, & (8a) \\ \nabla^2 \phi = 0, \quad \forall (X,Y) \in \mathbb{R}^2/\Omega_h, & (8b) \end{cases}$$

where Ω_h is the void space inside the crack.

Note that the coupling between mechanical and electrical variables still exists in boundary conditions. For permeable cracks, there is a nonzero electrical field in the free space inside the void, and the electrical potential inside the crack, $\tilde{\phi}$, satisfies the equation

$$\nabla^2 \tilde{\phi} = 0, \quad \mathbf{x} \in \Omega_h \tag{9}$$

which interacts with both the mechanical field as well as the electrical field outside the crack along the crack surfaces. To capture this interaction, one has to employ the exact boundary conditions of both continuum mechanics and electromagnetics to solve the crack problem.

Instead of imposing various combinations of boundary conditions to show the coupling between the primary variables and their conjugate pairs, only standard mixed boundary value problems are considered here (Malvern [30] and Jackson [31]). The boundary conditions or interface conditions for two different dielectric media are

- mechanical boundary conditions

$$\mathbf{n}\cdot[|\boldsymbol{\sigma}|] = -\hat{\mathbf{T}} \quad \text{on } S_\sigma; \quad \mathbf{u} = \hat{\mathbf{u}} \text{ on } S_u; \tag{10}$$

- electrical boundary conditions

$$\mathbf{n}\cdot[|\mathbf{D}|] = q_s \quad \text{on } S_D \quad \text{and} \quad \mathbf{n}\times[|\mathbf{E}|] = 0 \quad \text{on } S_E \tag{11}$$

where S_σ, S_u identify appropriate subsets of the domain boundary and $S = S_\sigma \cup S_u$. Note that the notation $[|f|] := f^+ - f^-$, and the normal vector $\mathbf{n}$ is pointing from medium$-$to medium$+$as shown in Fig. 1. In electrostatics, condition (11) can sometimes be replaced by the continuity condition of electric potential, i.e., $[|\phi|] = 0$. It should be noted that $S_\sigma \cap S_u = 0$, but $S_D \cap S_E \neq 0$.

In this paper, a planar permeable crack is modeled as a vanishing thin, finite dimension, rectangular-shaped slit with height $2h_0$ and width $2a$ as shown in Fig. 2.

As $h_0 \to 0$, the permeable crack becomes a conventional mathematical crack. One may write the crack height as the function of X,

$$h(X) = \begin{cases} h_0, & |X| < a \\ 0, & |X| > a. \end{cases} \tag{12}$$

The interior region of the crack is denoted as the set Ω_h,

$$\Omega_h := \{(X,Y) | -a < X < a, \quad \text{and} \quad -h_0 < Y < h_0\}. \tag{13}$$

Adjacent to the slit, there are two semi-infinite strips, which are denoted as Ω_s,

$$\Omega_s := \{(X,Y) | a < |X|, \quad \text{and} \quad -h_0 < Y < h_0\}. \tag{14}$$

3 Crack Solution

Consider a mode III permeable crack that is perpendicular to the poling direction (out plane), and it is subjected to remote traction and charge distribution at remote boundary (see Fig. 2).

Let $\hat{\mathbf{T}} = \tau_\infty \mathbf{e}_Y$ and $q_s = -q_\infty$.

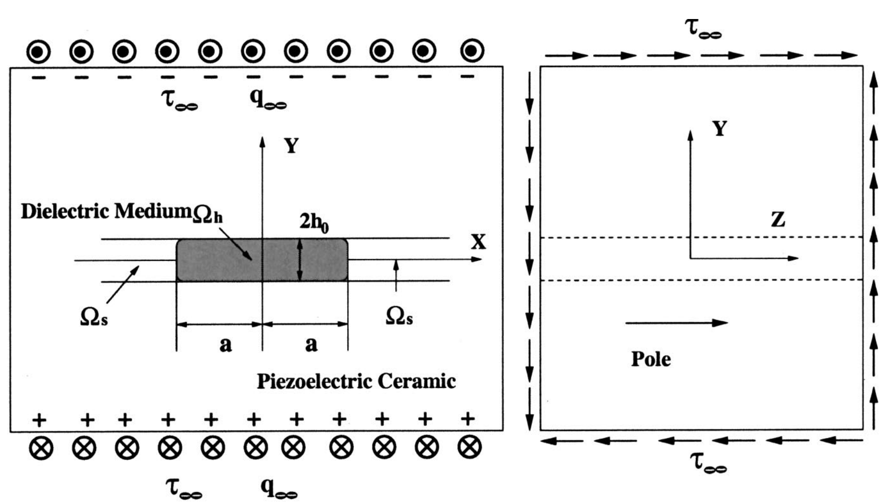

Fig. 2 A permeable crack with remote traction and charge distribution and surface charge distribution at the corner of the crack

$$\mathbf{n}\cdot[|\boldsymbol{\sigma}|]=-\hat{\mathbf{T}}\rightarrow\sigma_{YZ}=\tau_\infty,\quad \forall Y\rightarrow\infty \tag{15}$$

$$\mathbf{n}\cdot[|\mathbf{D}|]=q_s\rightarrow D_Y=q_\infty,\quad \forall Y\rightarrow\infty \tag{16}$$

where $q_s=-q_\infty$.

The boundary conditions on the crack surfaces

$$\mathbf{n}\cdot[|\boldsymbol{\sigma}|]=0,\quad \forall Y=\pm h_0 \quad \text{and} \quad |X|\leqslant a \tag{17}$$

$$\mathbf{n}\cdot[|\mathbf{D}|]=q_s,\quad \forall Y=\pm h_0 \quad \text{and} \quad |X|\leqslant a \tag{18}$$

$$\mathbf{n}\times[|\mathbf{E}|]=0,\quad \forall Y=\pm h_0 \quad \text{and} \quad |X|\leqslant a \tag{19}$$

take the form

$$\sigma_{YZ}(X,\pm h_0)=0,\quad \forall|X|\leqslant a \tag{20}$$

$$D_Y(X,\pm h_0)-D_Y^a(X,\pm h_0)=0,\quad \forall|X|\leqslant a \tag{21}$$

$$E_X(X,\pm h_0)-E_X^a(X,\pm h_0)=0,\quad \forall|X|\leqslant a. \tag{22}$$

The following symmetry conditions will be useful as well,

$$w(X,0)=0,\quad \forall|X|>a \tag{23}$$

$$\phi(X,0)=0,\quad \forall|X|>a \tag{24}$$

$$\phi^a(X,0)=0,\quad \forall 0<|X|<a \tag{25}$$

or

$$E_X(X,0)=0,\quad \forall|X|>a \tag{26}$$

$$E_X^a(X,0)=0,\quad \forall 0<|X|<a. \tag{27}$$

In the dielectric medium inside the crack, $D_i^a=\epsilon_0 E_i^a$ and $E_i^a=-\phi^a_{,i}$, $i=X,Y$.

Separate the displacement and electric potential fields into two parts: a uniform part due to the remote boundary conditions and a disturbance part due to the presence of the crack.

$$w=w_0+\tilde{w} \tag{28}$$

$$\phi=\phi_0+\tilde{\phi} \tag{29}$$

and choose

$$w_0=\gamma_\infty Y,\quad \phi_0=-E_\infty Y \tag{30}$$

and

$$\sigma_\infty=c_{44}^E\gamma_\infty-e_{15}E_\infty \tag{31}$$

$$q_\infty=e_{15}\gamma_\infty+\epsilon_{11}^s E_\infty \tag{32}$$

such that $\tilde{w}$, $\tilde{\phi}\rightarrow 0$ as $Y\rightarrow\infty$.

It is convenient to write the inverse relationship among key physical variables on the remote boundary,

$$\gamma_\infty=\frac{1}{\Delta_i}(\epsilon_{11}^S\tau_\infty+e_{15}q_\infty) \tag{33}$$

$$E_\infty=\frac{1}{\Delta_i}(-e_{15}\tau_\infty+c_{44}^E q_\infty), \tag{34}$$

where $\Delta_i:=c_{44}^E\epsilon_{11}^S+e_{15}^2$.

Extend the definition domain of ϕ^a into $\Omega_h\cup\Omega_s$ and let

$$\tilde{\phi}^a=\begin{cases}\phi^a-\phi_0^a, & \forall(X,Y)\in\Omega_h\\ 0, & \forall(X,Y)\in\Omega_s\end{cases} \tag{35}$$

where the uniform part of the electric potential is the leaky mode, which is chosen as $\phi_0^a:=-q_\infty/\epsilon_0 Y$.

Introduce the Fourier cosine transform

$$\begin{cases}F^*(\zeta,Y)=\sqrt{\dfrac{2}{\pi}}\displaystyle\int_0^\infty F(X,Y)\cos(\zeta X)dX\\ F(X,Y)=\sqrt{\dfrac{2}{\pi}}\displaystyle\int_0^\infty F^*(\zeta,Y)\cos(\zeta X)d\zeta\end{cases} \tag{36}$$

where $F(X,Y)=\tilde{w}(X,Y)$, $\tilde{\phi}(X,Y)$, and $\tilde{\phi}^a(X,Y)$, and $F^*(\zeta,Y)=\tilde{w}^*(\zeta,Y)$, $\tilde{\phi}^*(\zeta,Y)$, and $\tilde{\phi}^{a*}(\zeta,Y)$.

The transformed governing equations become

$$\frac{d^2}{dY^2}F^*+\zeta^2F^*=0. \tag{37}$$

Within the piezoelectric ceramic,

$$\tilde{w}^*(\zeta,Y)=A(\zeta)\exp(-\zeta Y),\quad \forall Y>0 \tag{38}$$

$$\tilde{\phi}^*(\zeta,Y)=B(\zeta)\exp(-\zeta Y),\quad \forall Y>0. \tag{39}$$

Inside the permeable crack,

$$\tilde{\phi}^{a*}(\zeta,Y)=C(\zeta)\sinh(\zeta Y),\quad \forall Y>0 \tag{40}$$

which satisfies the symmetry condition $\tilde{\phi}^a(X,0)=0$.

Consider the boundary condition

$$E_X(X,\pm h_0)-E_X^a(X,\pm h_0)=0,\quad |X|<a \tag{41}$$

and the symmetry condition

$$E_X(X,0)=0,\quad |X|>a, \tag{42}$$

and in the extended domain

$$\tilde{E}_X^a(X,0)=0,\quad |X|>a. \tag{43}$$

Combining Eqs. (41)–(43), one may find that

$$\tilde{E}_X(X,\pm h(X))-\tilde{E}_X^a(X,\pm h(X))=0,\quad \forall-\infty<X<+\infty \tag{44}$$

where function $h(X)$ is defined in Eq. (12).

In transformed space (ζ,Y), the condition (44) reads as

$$\tilde{E}_X^*(\zeta,\pm h^*(\zeta))-\tilde{E}_X^{a*}(\zeta,\pm h^*(\zeta))=0,\quad \forall 0<\zeta<+\infty \tag{45}$$

where

$$h^*(\zeta)=h_0\frac{\sin(a\zeta)}{\zeta}. \tag{46}$$

Considering Eqs. (39) and (40), one has

$$\begin{aligned}B(\zeta)&=C(\zeta)\,\frac{1}{2}(\exp(2\zeta h^*(\zeta))-1)\\ &=C(\zeta)\left(h_0\sin(a\zeta)+h_0^2\sin^2(a\zeta)+\frac{2}{3}h_0^3\sin^3(a\zeta)+\ \dots\right).\end{aligned} \tag{47}$$

Let

$$A(\zeta)=A_1(\zeta)+h_0A_2(\zeta)+h_0^2A_3(\zeta)+\ \dots \tag{48}$$

$$B(\zeta)=B_1(\zeta)+h_0B_2(\zeta)+h_0^2B_3(\zeta)+\ \dots\ . \tag{49}$$

By virtue of Eq. (47),

$$B_1(\zeta)=C(\zeta)h_0\sin(a\zeta) \tag{50}$$

$$B_2(\zeta)=C(\zeta)h_0\sin^2(a\zeta) \tag{51}$$

$$B_3(\zeta)=C(\zeta)\,\frac{2h_0}{3}\sin^3(a\zeta) \tag{52}$$

$$\dots\ . \tag{53}$$

After the Fourier transform, the boundary condition (21) becomes

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \zeta\{[e_{15}A(\zeta)-\epsilon_{11}^S B(\zeta)]\exp(-h_0\zeta) - \epsilon_0 C(\zeta)\cosh(\zeta h_0)\}\cos(\zeta X)d\zeta = 0, \quad \forall 0<X<a. \tag{54}$$

Note the subtlety in terms of crack surface position between Eq. (45) and Eq. (54). In the physical plane, the upper crack surface is at $Y=h_0$ for $|X|<a$, whereas in the transformed plane, $Y=h^*(\zeta)$, $0<\zeta<\infty$.

Consider the series expansion

$$[e_{15}A(\zeta)-\epsilon_{11}^S B(\zeta)] = [e_{15}A_1(\zeta)-\epsilon_{11}^S B_1(\zeta)] + h_0[e_{15}A_2(\zeta) - \epsilon_{11}^S B_2(\zeta)] + h_0^2[e_{15}A_3(\zeta)-\epsilon_{11}^S B_3(\zeta)] + \ldots \tag{55}$$

$$\exp(-h_0\zeta) = 1 - h_0\zeta + \frac{(h_0\zeta)^2}{2!} - \frac{(h_0\zeta)^3}{3!} + \ldots \tag{56}$$

$$\cosh(h_0\zeta) = 1 + \frac{(h_0\zeta)^2}{2!} + \ldots . \tag{57}$$

Assume that the permittivity constant, ϵ_0, is very small and comparable to h_0. The following asymptotic series integral equations may be derived:

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \zeta\left\{e_{15}A_1(\zeta) - \left(\epsilon_{11}^S + \frac{\epsilon_0}{h_0\sin(a\zeta)}\right)B_1(\zeta)\right\}\cos(\zeta X)d\zeta = 0, \quad \forall 0<X<a \tag{58}$$

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \{-\zeta^2([e_{15}A_1(\zeta)-\epsilon_{11}^S B_1(\zeta)] + \zeta[e_{15}A_2(\zeta) - \epsilon_{11}^S B_2(\zeta)]\}\cos(\zeta X)d\zeta = 0, \quad \forall 0<X<a \tag{59}$$

$$\ldots . \tag{60}$$

In the remainder of this paper, only the first-order approximation is considered. Moreover, when $h_0 \to 0$, $\sin(a\zeta)$ is always bounded. To render a tractable solution, we adopt the following average approximation:

$$h_0\sin(a\zeta) \approx h_0\overline{\sin(a\zeta)} \to 0 \tag{61}$$

where

$$\overline{\sin(a\zeta)} := \sqrt{\frac{\pi}{2}}\int_0^\infty \sin(a\zeta)d\zeta = \sqrt{\frac{\pi}{2}}\frac{1}{a}. \tag{62}$$

The identity (62) is in the sense of a generalized function (see Erdélyi et al. [32] or Lighthill [33], p. 33).

Let

$$r := \sqrt{\frac{2}{\pi}}\frac{a}{h_0}. \tag{63}$$

Equation (54) becomes

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \zeta(e_{15}A_1(\zeta) - (\epsilon_{11}^S + \epsilon_0 r)B_1(\zeta))\cos(\zeta X)d\zeta = 0, \quad \forall 0<X<a. \tag{64}$$

The first-order approximation of boundary condition (20) provides the additional integral equation

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \zeta(c_{44}^E A_1(\zeta) + e_{15}B_1(\zeta))\cos(\zeta X)d\zeta = \tau_\infty, \quad \forall 0<X<a. \tag{65}$$

Considering the symmetry conditions $w(X,0)=\phi(X,0)=0$, $\forall |X|>a$. Two sets of standard dual integral equations may be derived;

$$\begin{cases} \sqrt{\frac{2}{\pi}}\int_0^\infty \zeta A_1(\zeta)\cos(\zeta X)d\zeta = S, & |X|<a \\ \int_0^\infty A_1(\zeta)\cos(\zeta X)d\zeta = 0, & |X|>a \end{cases} \tag{66}$$

and

$$\begin{cases} \sqrt{\frac{2}{\pi}}\int_0^\infty \zeta B_1(\zeta)\cos(\zeta X)d\zeta = T, & |X|<a \\ \int_0^\infty B_1(\zeta)\cos(\zeta X)d\zeta = 0, & |X|>a \end{cases} \tag{67}$$

where

$$S := \frac{(\epsilon_{11}^S + \epsilon_0 r)\tau_\infty}{\Delta} \tag{68}$$

$$T := \frac{e_{15}\tau_\infty}{\Delta} \tag{69}$$

and $\Delta = c_{44}^E(\epsilon_{11}^S + \epsilon_0 r) + e_{15}^2$.

Let

$$A_1(\zeta) = \sqrt{\frac{\pi}{2}}\frac{Sa}{\zeta}J_1(a\zeta) \tag{70}$$

$$B_1(\zeta) = \sqrt{\frac{\pi}{2}}\frac{Ta}{\zeta}J_1(a\zeta). \tag{71}$$

Consequently, one may find that

$$w(X,Y) = \gamma_\infty Y + \frac{(\epsilon_{11}^S+\epsilon_0 r)\tau_\infty}{\Delta}a\int_0^\infty \zeta^{-1}J_1(a\zeta)\cos(\zeta X) \times \exp(-\zeta Y)d\zeta \tag{72}$$

$$\phi(X,Y) = -E_\infty Y + \frac{e_{15}\tau_\infty}{\Delta}a\int_0^\infty \zeta^{-1}J_1(a\zeta)\cos(\zeta X)\exp(-\zeta Y)d\zeta \tag{73}$$

and

$$w(X,0) = \frac{(\epsilon_{11}^S+\epsilon_0 r)\tau_\infty}{\Delta}\begin{cases}\sqrt{a^2-X^2}, & |X|<a \\ 0, & |X|>a\end{cases} \tag{74}$$

$$\phi(X,0) = \frac{e_{15}\tau_\infty}{\Delta}\begin{cases}\sqrt{a^2-X^2}, & |X|<a \\ 0, & |X|>a\end{cases}. \tag{75}$$

4 Intensity Factors

Let $Y=0$. The asymptotic fields of both mechanical and electric variables in front of the crack tip are found as follows:

$$\epsilon_{YZ} = \frac{(\epsilon+\epsilon_0 r)\tau_\infty}{\Delta}\frac{X}{\sqrt{X^2-a^2}} + \left(\gamma_\infty - \frac{(\epsilon+\epsilon_0 r)\tau_\infty}{\Delta}\right) + \text{higher order terms} \tag{76}$$

$$E_Y = -\frac{e_{15}\tau_\infty}{\Delta}\frac{X}{\sqrt{X^2-a^2}} + \left(E_\infty + \frac{e_{15}\tau_\infty}{\Delta}\right) + \text{higher order terms} \tag{77}$$

$$\sigma_{YZ} = \frac{\tau_\infty X}{\sqrt{X^2-a^2}} + \text{higher order terms} \tag{78}$$

$$D_Z=\frac{e_{15}\epsilon_0 r\tau_\infty}{\Delta}\frac{X}{\sqrt{X^2-a^2}}+\left(q_\infty-\frac{e_{15}\epsilon_0 r\tau_\infty}{\Delta}\right)$$
$$+\text{higher order terms.} \quad (79)$$

The relevant field intensity factors can be found as follows:

$$K_{III}^{\gamma}=\lim_{X\to a^+}\sqrt{2\pi(X-a)}\epsilon_{YZ}(X,0)=\frac{(\epsilon_{11}^S+\epsilon_0 r)\tau_\infty\sqrt{\pi a}}{\Delta} \quad (80)$$

$$K_{III}^{E}=\lim_{X\to a^+}\sqrt{2\pi(X-a)}E_Y(X,0)=-\frac{e_{15}\tau_\infty\sqrt{\pi a}}{\Delta} \quad (81)$$

$$K_{III}^{\tau}=\lim_{X\to a^+}\sqrt{2\pi(X-a)}\sigma_{YY}(X,0)=\tau_\infty\sqrt{\pi a} \quad (82)$$

$$K_{I}^{D}=\lim_{X\to a^+}\sqrt{2\pi(X-a)}D_Y(X,0)=\frac{e_{15}\epsilon_0 r\tau_\infty\sqrt{\pi a}}{\Delta}. \quad (83)$$

Assume that the permittivity inside the crack is very small, $\epsilon_0 \ll h_0$, or $\epsilon_0\to 0$, we may recover all the results obtained by Zhang and Hack [17] for a mode III crack.

$$K_{III}^{\gamma}=\frac{\epsilon}{\Delta_i}\tau_\infty\sqrt{\pi a} \quad (84)$$

$$K_{III}^{E}==-\frac{e_{15}}{\Delta_i}\tau_\infty\sqrt{\pi a} \quad (85)$$

$$K_{III}^{\tau}=\tau_\infty\sqrt{\pi a} \quad (86)$$

$$K_{III}^{D}=0 \quad (87)$$

Let $h_0=0$ and consequently $r\to\infty$. That is, the slit has zero height. The physical interpretation of this limit is that the upper and lower crack surfaces are constantly in close contact during fracture process, there is no dielectric medium inside the crack. The intensity factors become

$$K_{III}^{\gamma}=\frac{\epsilon_{11}^S}{c_{44}^E}\tau_\infty\sqrt{\pi a} \quad (88)$$

$$K_{III}^{E}=0 \quad (89)$$

$$K_{III}^{\tau}=\tau_\infty\sqrt{\pi a} \quad (90)$$

$$K_{III}^{D}=0. \quad (91)$$

This recovers the solution obtained by Yang and Kao [34] for a zero-height crack in piezoelectric medium.

5 Energy Release Rate

It is generally believed that energy release rate, or J-integral, is a better criterion for crack growth than stress intensity factors. The J-integral in a piezoelectric medium is given by Cherepanov [35],

$$J=\int_\Gamma (Hn_1-\sigma_{ij}n_i u_{j,1}-n_i D_i\phi_{,1})dS \quad (92)$$

where H is the electric enthalpy density.

On the surface of a permeable crack, both the normal component of electric displacement as well as the electric potential are not zero, consequently, the contribution in the contour integral, J, along crack surfaces is not zero. Therefore, for permeable cracks, two types of J-integrals can be defined: *local energy release rate* and *global energy release rate*. The global energy release rate consists of two parts: (1) *local energy release rate* and (2) the energy release rate due to interaction between dielectric medium inside the crack and piezoelectric matrix along crack surfaces. The local energy release rate is defined as the contour integral, J, along an infinitesimal circle around the crack tip, Γ_l. The global energy release rate may be defined as any contour integral, J, starting at the center of the lower part of the crack surface and ending at the center of upper part of the crack surface (see Fig. 3). Therefore, the global energy release rate is the sum of the local energy release rate and the contour integral contribution along the crack surfaces, i.e.,

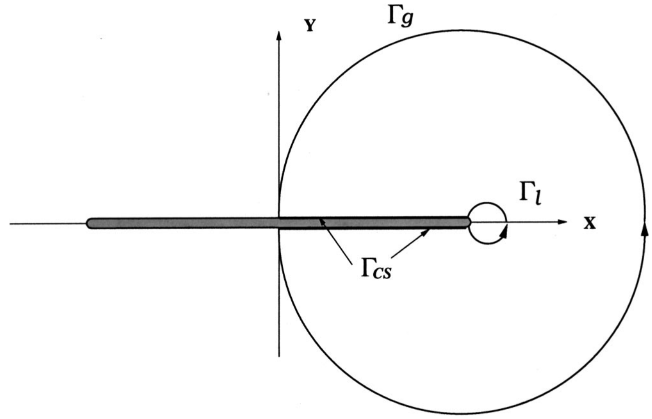

Fig. 3 *J*-integral contours for evaluating local and global energy release rates

$$J_g=J_l+J_{cs} \quad (93)$$

where J_{cs} denote the energy release rate contribution from crack surfaces, which can be calculated by

$$J_{cs}=-\int_{cs} n_i D_i\phi_{,x}dS. \quad (94)$$

5.1 Local Energy Release Rate. We first consider the so-called *local energy release rate*. Consider the following electromechanical boundary conditions:

$$\sigma_{YY}=\tau_\infty,\quad D_Y=q_\infty,\quad \forall Y\to\infty. \quad (95)$$

The corresponding local energy release rate of the present permeable crack model is

$$J_l^{NEW}=\frac{1}{2}(K_{III}^{\tau}K_{III}^{\gamma}-K_{III}^{E}K_{III}^{D})=\frac{\pi a}{2}\frac{\tau_\infty^2}{\Delta^2}(\Delta(\epsilon_{11}^S+\epsilon_0)+e_{15}^2\epsilon_0 r). \quad (96)$$

Letting $\epsilon_0=0$ in (96), one recovers the result obtained by Zhang and Hack [17], i.e.,

$$J_l^{NEW}\Rightarrow\frac{\pi a}{2}\frac{\epsilon_{11}^S}{\Delta_i}\tau_\infty^2. \quad (97)$$

Let $h_0=0$ or $r\to\infty$ in Eq. (96). The result obtained by Yang and Kao [34] may be recovered,

$$J_l^{NEW}\Rightarrow\frac{\pi a}{2}\frac{\tau_\infty^2}{c_{44}^E}. \quad (98)$$

Equation (98) is the purely elastic energy release rate, since there is no dielectric medium inside the crack.

5.2 Global Energy Release Rate. When a permeable crack grows, energy release is not only consumed in supplying the surface energy for newly formed crack surfaces, but also consumed by supplying the electrostatic energy to the dielectric medium inside the crack. In fact, if the surface charge is absent on the crack surfaces, the normal component of electric displacement in piezoelectric medium may be equal to the normal component of electric displacement in the dielectric medium inside the crack. This suggests that the crack surface contribution to the J-integral is the part of energy release rate that goes directly into supplying the electrostatic energy increase in the dielectric medium inside the crack.

If the surface charge is present on crack surfaces, which may either enhance or reverse the direction of the energy-moment flux, an additional energy release rate may be created that will influence crack growth process.

In order the evaluate J_g, we first evaluate J_{cs}. Consider the normal component of the electric displacement on the crack surfaces.

$$\begin{aligned} D_Y(X,h_0) &\approx D_y(X,0) \\ &= e_{15}\frac{\partial w}{\partial Y} - \epsilon_{11}^S \frac{\partial \phi}{\partial Y} \\ &= q_\infty - \frac{e_{15}\epsilon_0 r \tau_\infty}{\Delta} a \int_0^\infty J_1(a\zeta)\cos(\zeta X)\, d\zeta \\ &= q_\infty - \frac{e_{15}\epsilon_0 r \tau_\infty}{\Delta} \begin{cases} 1, & |X|<a \\ 1-\dfrac{X}{\sqrt{X^2-a^2}}, & |X|>a \end{cases}. \end{aligned} \tag{99}$$

Substituting Eq. (99) and Eq. (75) into Eq. (94) yields

$$\begin{aligned} J_{cs} &= D_Y(0,0^+)(\phi(0,0^+)-\phi(0,0^-)) \\ &= \left(q_\infty - \frac{\epsilon_0 e_{15} r \tau_\infty}{\Delta}\right)\left(\frac{2e_{15}\tau_\infty a}{\Delta}\right). \end{aligned} \tag{100}$$

Hence the global energy release has the form

$$\begin{aligned} \text{I:} \quad J_{cr}^g = \left(\frac{\pi a}{2\Delta}\right)\Bigg\{ \left[(\epsilon_{11}^S + \epsilon_0 r) + \frac{e_{15}^2 \epsilon_0 r}{\Delta}\left(1-\frac{4}{\pi}\right)\right]\tau_\infty^2 \\ + \frac{4}{\pi} e_{15}\tau_\infty q_\infty \Bigg\}. \end{aligned} \tag{101}$$

Let $\epsilon_0 = 0$. The global energy release rate becomes

$$\text{II:} \quad J_{cr1}^g = \left(\frac{\pi a}{2c_{44}^E}\right)\left(\frac{c_{44}^E \epsilon_{11}^S + \frac{4}{\pi} e_{15}^2}{c_{44}^E \epsilon_{11}^S + e_{15}^2}\tau_\infty^2 + \frac{4}{\pi} e_{15}^2 \tau_\infty E_\infty\right). \tag{102}$$

If $h_0 \to 0$, the global energy release rate becomes

$$J_{cr2}^g = \left(\frac{\pi a}{2}\right)\frac{\tau_\infty^2}{c_{44}^E} \tag{103}$$

which was previously found by Yang and Kao [34].

Since $4/\pi = 1.273238 \approx 1.0$, the newly derived results (101) and (102) are very close to the empirical result proposed by Park and Sun [11,12].

$$J_{PS} = \frac{\pi a}{2(c_{44}^E \epsilon_{11}^S + e_{15}^2)}(\epsilon_{11}^S \tau_\infty^2 + e_{15}\tau_\infty q_\infty) \tag{104}$$

This result also agrees with the result obtained by Mao et al. [36] in analyzing a mode I crack by considering toughening under polarization switching.

6 Closure

The analysis presented in this work reveals that the interaction between a crack and its permeable environment can be crucial to crack growth in a piezoelectric ceramic. This interaction may be quantified through a *J*-integral along permeable crack surfaces. A global energy release rate that taking into account this effect may serve better as the fracture toughness for piezoelectric ceramics.

It has been an outstanding problem regarding the energy release rate of a piezoelectric crack. The impermeable crack solution always gives the negative energy release rate, presenting a false impression that the applied electric field will prohibit crack growth. The fallacy of impermeable approximation is that it shields, and may even reverse the direction of energy-momentum flux on the crack surface. The permeable crack model presented in this paper provides a leaky mode for an electrical field, allowing the applied electric field pass through the dielectric medium inside the crack. An in-depth analysis for a mode I permeable crack is presented in a recent paper by Li [37].

Based on the asymptotic analysis, a first-order approximation solution is obtained for a mode III crack in a permeable environment. The control parameters of the asymptotic analysis are the crack height, h_0, dielectric permittivity inside the crack, ϵ_0, and the crack width, a.

It has been found that the global energy release rate derived for a permeable crack is in broad agreement with the known experimental observations (e.g., [11,12]), which is in contrast with the local energy release rate criterion proposed by Gao et al. [13,14] according to the saturation-strip model. Nevertheless, for all practical purposes, it may be a good estimate that

$$J_l < J < J_g, \quad \text{or} \quad J_g < J < J_l \tag{105}$$

since the actual contour integral may has a path between Γ_l and Γ_g (see Fig. 3).

The global energy release rate derived here may be served as a fracture criterion for piezoelectric materials in general. This contribution reconciles the discrepancy between experimental observations and theoretic analyses without invoking any nonlinear theory, and it explains, by rigorous analysis, how an applied electric field affects crack growth in a piezoelectric ceramic through its interaction with the permeable environment surrounding the crack.

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