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On the micromechanics theory of Reissner-Mindlin plates

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Summary. A micromechanics model is developed for the Reissner-Mindlin plate. A generalized eigenstrain formulation, i.e., an eigencurvature/eigen-rotation formulation, is proposed, which is the analogue or counterpart of the eigenstrain formulation in linear elasticity. The micromechanics model of the Reissner-Mindlin plate is useful in the study of mechanical behavior of composite plates that contain randomly distributed inhomogeneities, whose sizes are close to the order of thickness of the plate; under those circumstances, the use of micromechanics of linear elasticity is not justified, and, moreover, it is inconsistent with structural theories, such as the Reissner-Mindlin plate theory, that are actually used in engineering design.

In this paper, the analytical solution of an elliptical inclusion embedded in an infinite thick plate is sought. In particular, the first order asymptotic (or approximated) solution of the elliptical inclusion problem is obtained in explicit form. Accordingly, the Eshelby tensors of the Reissner-Mindlin plate are derived, which relate eigencurvature and eigen-rotation to the induced curvature and shear deformation fields. Several variational inequalities of the Reissner-Mindlin plate are discussed and derived, including the comparison variational principles of Hashin-Shtrikman/Talbot-Willis type. As an application, variational bounds are derived to estimate the effective elastic stiffness of Reissner-Mindlin plates, specifically, the flexural rigidity and transverse shear modulus. The newly derived bounds are congruous with the Reissner-Mindlin plate theory, and they provide an optimal estimation on effective rigidity as well as effective transverse shear modulus for unstructured composite thick plates.

1 Introduction

In this paper, we are concerned with a micromechanics model of Reissner-Mindlin plates, which is an important subject in engineering practices, and, to the author's knowledge, it has been neglected both in mechanics literature and engineering design. Part of the reason for such oblivion is that in the past the term, "*composite plate*", is only referred to the multiphase plates that have definite structures, such as laminar plates, or well structured lattice plates (e.g., Christensen [4], Mindlin [28], and Kaprielion et al. [20]). Today, many composite plates are made of materials with randomly distributed heterogeneous constituents or unstructured composite materials, and therefore such "oblivion" becomes inexcusable. Recently, a micromechanics model of the Love-Kirchhoff plate is proposed by Li [26]. This study is a further development of micromechanics in the framework of structural mechanics, in an attempt to compound a systematic exposition of structural micromechanics, which is in parallel with the micromechanics in linear elasticity.

The micromechanics of linear elasticity theory rests upon the notion of *representative volume element* (RVE) (Hill [16], Hashin [15], Kröner [22], [23], Willis [49], and Nemat-Nasser and Hori [32]), which is essentially a provision on the length scale of the aggregates, within

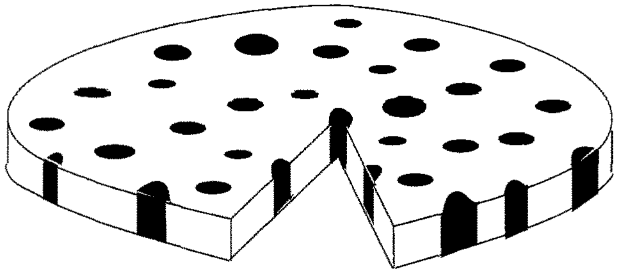


Fig. 1. A representative area element of a Reissner-Mindlin plate made by unstructured composite materials

which the micromechanics theory is valid in a statistical sense. For a composite plate, in which that characteristic size of the micro-element is only one order magnitude less than or at the same order of the thickness of the plate, the application of conventional micro-elasticity will become questionable. Furthermore, since the conventional micro-elasticity theory does not carry any information about structural mechanics, if a structure's effective stiffness, such as flexural rigidity or transverse shear modulus, is evaluated by linear micro-elasticity, it is certainly not compatible with the theory of structure mechanics that is used in actual strength analysis. Strictly speaking, the constitutive equations of Reissner-Mindlin plates, i.e., the relations between moment/resultant and curvature/rotation, are derived by taking into account additional internal constraints on the constitutive laws of linear elastic materials (e.g., Naghdi [30], [31]). Thus, the constitutive relations at the structural level are fundamentally different from the constitutive relations at continuum level. For instance, a Reissner-Mindlin plate is intrinsically anisotropic in a mathematical sense, even if the matrix material of the plate is isotropic. It is, therefore, erroneous in principle to evaluate the effective stiffness of the Reissner-Mindlin plate based on the micromechanics of linear elasticity theory.

From this standpoint, developing micromechanics models in structure mechanics can be instrumental in the engineering design analysis. To pursue such a novel scheme, the first logical step seems to be replacing the notion of the representative volume element by the notion of the representative area element (RAE): An representative area element defined for a material point in a two-dimensional (2-D) manifold is a material element which is a statistically representative of all the material points in a material neighborhood at a specified scale on the two-dimensional manifold. The continuum material point is called *meso-area-element*, whereas its micro-constituents are called *micro-area-elements*. An RAE must include a very large number of micro-area-elements, in order that the representative information is statistically stable. However, in three-dimensional (3-D) space, there is no physical object that is truly a 2-D mathematical manifold. An representative area element is actually a special representative volume element whose properties are homogeneous, in an average sense, in the direction that is perpendicular to the surface area of every micro-area-element. For a Reissner-Mindlin plate, it implies that the material concentration of every micro-area-element dominates statistically in the thickness direction. This is consistent with the Reissner-Mindlin plate theory, because the theory of Reissner-Mindlin plates is constructed through a special "homogenization" process (or averaging process) of 3-D elasticity theory in the thickness direction of the plate. Figure 1 illustrates an ideal model of such a representative area element. In reality, the inhomogeneous phases do not need to penetrate through the thickness of the plate, since it is only required that the concentrations of every species dominate *statistically* in the thickness direction.

In Reissner-Mindlin plate theory, two rotational degrees of freedom are assigned at each material point on the plate's middle surface, and the rotation vector lies on the plane of the middle surface, which makes the Reissner-Mindlin plate a prototype of a Cosserat surface in solid mechanics (Green and Naghdi [8])[1]. It is well known that the nature of a Cosserat continuum is fairly different from the nonpolar continuum, and it brings the "strain gradient" effect into the picture. Usually it may introduce an intrinsic length scale into the continuum theory (see [10]), which creates some intricacies in the mathematical treatment. For instance, in the elliptical inclusion solution, the prescribed constant eigen-curvature and eigen-rotation may not induce a constant curvature field as well as a constant rotation field inside the elliptical inclusion. To further complicate the situation, there is a interaction between the curvature field and the rotation field. Therefore, it is a non-trivial task to extend micromechanics to the Reissner-Mindlin plate.

The paper is organized as follows: in Sect. 2, a detailed solution procedure of an elliptical inclusion problem is presented, and the Reissner-Mindlin plate version of Eshelby tensors, which relate the first-order asymptotic solution to the prescribed eigen-curvature/eigen-rotation field, is documented. Section 3 is devoted to a variational treatment of Reissner-Mindlin plates; in passing, the averaging properties of the Reissner-Mindlin plate are studied in detail. Subsequently, in Sect. 4, the new variational bounds on the effective stiffness for Reissner-Mindlin plates are derived explicitly, and several examples are discussed in connection to applications.

It should be noted that the idea of seeking the effective elastic stiffness of thin plates as well as thick plates is not new[2]. Qin et al. [34] appear to be the first ones to use the concept of generalized eigen-strain in classical plate theory. Prior to that, Caillerie [2] used homogenization techniques to analyze both non-homogeneous thin and thick plates based on the linear elasticity theory; however, its engineering significance has yet to be explored. In fact, various theories of laminar plates are essentially discrete homogenizations in the thickness direction of the plate. To complement the laminar plate theories, it is an inevitable course to search for a micromechanics theory within a low-dimensional Cosserat continuum.

2 Eigen-curvature and eigen-rotation formulation

2.1 Preliminaries

There are standard references on the theory of Reissner-Mindlin plates, such as the original papers by Reissner [35]–[37], Mindlin [27], and contemporary treatises, e.g., Constanda [5]. For easy reference, a brief list of formulas is given in the following.

Assume that Ω is a simply connected, bounded region in $\mathbb{R}^2$, which has a smooth boundary; subsequently, the normal vector, $\mathbf{n}$, and tangential vector, $\mathbf{s}$, on $\partial\Omega$ are uniquely defined. The material space of the plate is defined as $\Omega \times [-h/2, h/2] \subset \mathbb{R}^3$, (see Fig. 2), where h is the thickness of the plate that contains a composite material of multiple constituent phases. For convenience, we assume that the plate has n different phases, and each phase has distinct elastic moduli.

[1] There are also director theories proposed in fluid mechanics as well (e.g., Green and Naghdi [9], [11], [12]).

[2] In this paper, the term "*thick plate*" is used strictly as the synonym of a Reissner-Mindlin plate, whereas the term "*thin plate*" is used as the synonym of a Love-Kirchhoff plate.

As a priori condition, the following kinematic assumptions about the plate are assumed automatically fulfilled for the composite plate under consideration:

(i) The plate median surface does not stretch or contract;
(ii) There is no thickness stretch;
(iii) The plate deforms due to flexure as well as shear strain.

The deflection of the plate is defined as

$$w : \Omega \to \mathbb{R} . \tag{1}$$

The cross section of the plate is still assumed as rigid, but it can rotate independently from deflection. The rotation at each material point can be characterized by a vector, $\boldsymbol{\phi} = \phi_\alpha \mathbf{e}^\alpha$,

$$\boldsymbol{\phi} : \Omega \to \mathbb{R} \times \mathbb{R} . \tag{2}$$

The flexural curvature of the plate, and the shear deformation of the plate are defined as

$$\chi_{\alpha\beta} = \frac{1}{2} \left(\phi_{\alpha,\beta} + \phi_{\beta,\alpha} \right) , \tag{3}$$

$$\gamma_\alpha = \phi_\alpha + w_{,\alpha} , \tag{4}$$

where (and in the rest of the paper) the Greek index always ranges from 1 to 2. In Fig. 3, a pictorial illustration is given to describe the relationship between transverse shear deformation and the rotation of the cross section.

For Reissner-Mindlin plates, the general constitutive relations involve the vertical external load (e.g., Vander Weeën [41]). Since we are only interested in the intrinsic micromechanics properties on a Cosserat surface, the external load, q, is always assumed to be zero in this paper. Thus, the constitutive equations on the Cosserate surface are simplified as

$$m_{\alpha\beta} = L_{\alpha\beta\zeta\eta} \chi_{\zeta\eta} , \tag{5}$$

$$Q_\alpha = G_{\alpha\beta} \gamma_\beta \tag{6}$$

or, inversely,

$$\chi_{\alpha\beta} = N_{\alpha\beta\zeta\eta} m_{\zeta\eta} , \tag{7}$$

$$\gamma_\alpha = H_{\alpha\beta} Q_\beta , \tag{8}$$

where $L_{\alpha\beta\zeta\eta}$ is the elastic stiffness tensor of the plate, and $G_{\alpha\beta}$ are the transverse shear moduli; correspondingly, $N_{\alpha\beta\zeta\eta}$ is the elastic compliance tensor, and $H_{\alpha\beta}$ are the transverse shear compliances. For an isotropic plate, the elastic stiffness and elastic compliance take the forms

$$L_{\alpha\beta\zeta\eta} = \frac{D(1-\nu)}{2} \left(\delta_{\alpha\zeta}\delta_{\beta\eta} + \delta_{\alpha\eta}\delta_{\beta\zeta} \right) + D\nu d_{\alpha\beta}\delta_{\zeta\eta} , \tag{9}$$

$$G_{\alpha\beta} = D \frac{(1-\nu)}{2} \lambda^2 \delta_{\alpha\beta} = G\varkappa^2 h \delta_{\alpha\beta} = G_p \delta_{\alpha\beta} , \tag{10}$$

$$N_{\alpha\beta\zeta\eta} = \frac{(1+\nu)}{2D(1-\nu^2)} \left(\delta_{\alpha\zeta}\delta_{\beta\eta} + \delta_{\alpha\eta}\delta_{\beta\zeta} \right) - \frac{\nu}{D(1-\nu^2)} \delta_{\alpha\beta}\delta_{\zeta\eta} , \tag{11}$$

$$H_{\alpha\beta} = G_p^{-1} \delta_{\alpha\beta} , \tag{12}$$

where ν is the Poisson ratio, D is the flexural rigidity, $D := \dfrac{Eh^3}{12(1-\nu^2)}$, and G_p is the transverse shear modulus, $G_p := \dfrac{E\varkappa^2 h}{2(1+\nu)}$.

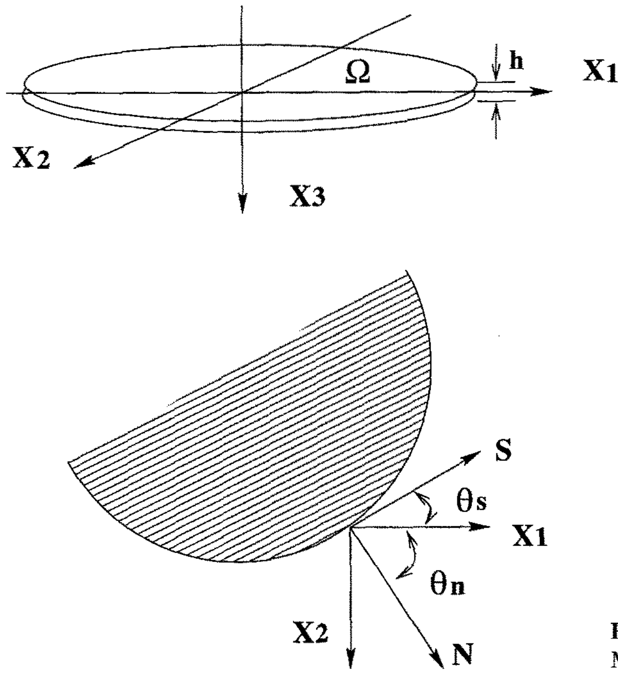


Fig. 2. The configuration of a Reissner-Mindlin plate

Remark 2.1 For a Reissner plate, $\lambda = \sqrt{10}/h$; for a Mindlin plate, $\lambda = \sqrt{12}\varkappa/h$. In general the shear coefficient, $\varkappa = \varkappa(\nu)$, is a function of Poisson's ratio, and it can be determined by considering dynamics effects (Mindlin [27]). Mindlin showed that $\varkappa^2$ is almost linearly dependent on Poisson's ratio, ν. It ranges from 0.76 for $\nu = 0$ to 0.91 for $\nu = 1/2$. In this paper, we adopt Mindlin's interpretation.

□

By taking all external loads as zero, the equilibrium equations of Reissner-Mindlin plates are:

$$m_{\alpha\beta,\beta} - Q_\alpha = 0\,, \tag{13}$$

$$Q_{\alpha,\alpha} = 0\,. \tag{14}$$

On $\partial\Omega = S_u \cup S_F$, two types of boundary conditions are posed:

– Deformation prescribed boundary conditions on S_u,

$$\phi_n := \phi_\gamma n_\gamma = \hat{\phi}_n\,, \tag{15}$$

$$\phi_s := \phi_\gamma s_\gamma = \hat{\phi}_s\,, \tag{16}$$

$$w = \hat{w}\,. \tag{17}$$

– Force prescribed boundary conditions on S_F,

$$M_n := m_{\alpha\beta} n_\alpha n_\beta = \hat{M}_n\,, \tag{18}$$

$$M_s := m_{\alpha\beta} n_\alpha s_\beta = \hat{M}_s\,, \tag{19}$$

$$Q_n := Q_\alpha n_\alpha = \hat{Q}_n\,. \tag{20}$$

Following the convention, we denote the rotation and deflection field, (ϕ_α, w), that satisfies the prescribed deformation boundary conditions (15)–(17) as the kinematically admissible

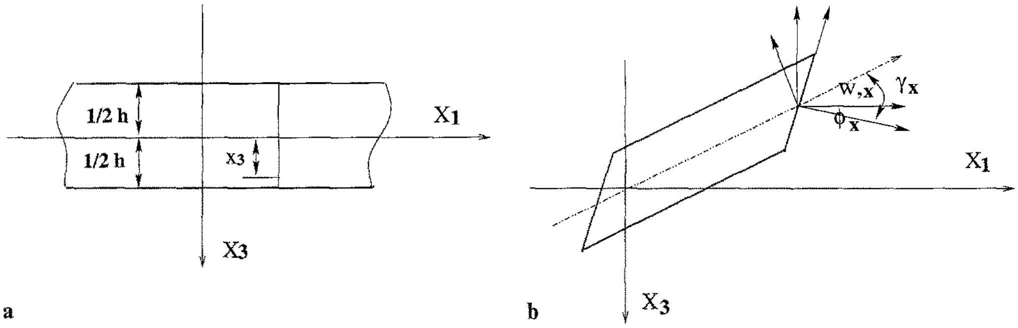


Fig. 3. An illustration of kinematic assumptions of the Reissner-Mindlin plate: **a** reference configuration; **b** deformed configuration

deformation field, whereas we denote the moment tensor and shear resultant field, $(m_{\alpha\beta}, Q_\alpha)$, that satisfies the prescribed force boundary conditions (18)–(20) and the equilibrium equations (13) and (14) as the statically admissible resultant field.

Lemma 2.1. Suppose $\chi_{\alpha\beta} = \frac{1}{2}(\phi_{\alpha,\beta} + \phi_{\beta,\alpha}), \gamma_\alpha = \phi_\alpha + w_{,\alpha}$ and $(\phi_\alpha, w) = 0$, $\mathbf{x} \in \partial\Omega$. Then the condition

$$\int\int_{\Omega} (m^0_{\alpha\beta}\chi_{\alpha\beta} + Q_\alpha{}^0\gamma_\alpha)\, d\Omega = 0 \tag{21}$$

implies that $\forall\ \mathbf{x} \in \Omega$:

$$m^0_{\alpha\beta,\beta} - Q_\alpha{}^0 = 0\,, \tag{22}$$

$$Q^0_{\alpha,\alpha} = 0\,. \tag{23}$$

Proof: Integration by parts yields

$$\begin{aligned}\int\int_{\Omega} (m^0_{\alpha\beta}\chi_{\alpha\beta} + Q_\alpha{}^0\gamma_\alpha)\, d\Omega &= \oint_{\partial\Omega} (m^0_{\alpha\beta} n_\beta \phi_\alpha + Q_\alpha{}^0 n_\alpha w)\, dS\,, \\ &\quad - \int\int_{\Omega} \{(m^0_{\alpha\beta,\beta} - Q_\alpha{}^0)\,\phi_\alpha + Q^0_{\alpha,\alpha} w\}\, d\Omega \\ &= - \int\int_{\Omega} \{(m^0_{\alpha\beta,\beta} - Q_\alpha{}^0)\,\psi_\alpha + Q^0_{\alpha,\alpha} w\}\, d\Omega\,.\end{aligned}$$

Then, (22) and (23) follow immediately. □

Lemma 2.2. Let $\chi_{\alpha\beta} \in C^2(\Omega)$, $\gamma_\alpha \in C^1(\Omega)$, $m_{\alpha\beta}, Q_\alpha \in C_0{}^1(\Omega)$, and $m_{\alpha\beta,\beta} - Q_\alpha = 0$, $Q_{\alpha,\alpha} = 0$. Then the condition

$$\int\int_{\Omega} (m_{\alpha\beta}\chi_{\alpha\beta} + Q_\alpha\gamma_\alpha)\, d\Omega = 0\,, \tag{24}$$

implies that

$$\frac{1}{2}\,[\varepsilon_{\alpha\eta}(\chi_{\alpha\beta,\eta} + \gamma_{\eta,\alpha\beta}) + \varepsilon_{\alpha\eta}\chi_{\beta\alpha,\eta}] = 0\,. \tag{25}$$

Note that $\varepsilon_{\alpha\beta}$ is the 2-D permutation symbol,

$$\varepsilon_{\alpha\beta} = \begin{cases} 1; & \alpha > \beta \\ 0; & \alpha = \beta\,. \\ -1; & \alpha < \beta \end{cases} \tag{26}$$

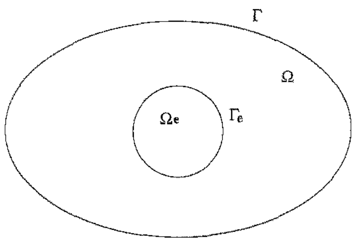


Fig. 4. An inclusion inside a matrix

Proof: Since $m_{\alpha\beta,\beta} - Q_\alpha = 0$ and $Q_{\alpha,\alpha} = 0 \;\; \forall\, \mathbf{x} \in \mathrm{int}\,\{\Omega\}$, the Reissner-Mindlin plate admits the following stress function representation:

$$m_{\alpha\beta} = \frac{1}{2}\left[\varepsilon_{\alpha\eta}\psi_{\beta,\eta} + \varepsilon_{\beta\eta}\psi_{\alpha,\eta}\right], \tag{27}$$

$$Q_\alpha = \frac{1}{2}\,\varepsilon_{\alpha\eta}\psi_{\gamma,\gamma\eta}, \tag{28}$$

where $\psi \in C_0{}^2(\Omega)$. Then, by the Gauss theorem and integration by parts,

$$\begin{aligned}\int\!\!\int_\Omega (m_{\alpha\beta}\chi_{\alpha\beta} + Q_\alpha\gamma_\alpha)\,d\Omega &= \frac{1}{2}\int\!\!\int_\Omega \left(\left[\varepsilon_{\alpha\eta}\psi_{\beta,\eta} + \varepsilon_{\beta\eta}\psi_{\alpha,\eta}\right]\chi_{\alpha\beta} + \varepsilon_{\alpha\eta}\psi_{\gamma,\gamma\eta}\gamma_\alpha\right) d\Omega\,,\\ &= \frac{1}{2}\oint_{\partial\Omega} \left\{\left(\left[\varepsilon_{\alpha\eta}\psi_\beta + \varepsilon_{\beta\eta}\psi_\alpha\right]\chi_{\alpha\beta} + \varepsilon_{\alpha\eta}\psi_{\gamma,\gamma}\gamma_\alpha\right)n_\eta - \varepsilon_{\alpha\eta}\psi_\gamma n_\gamma\gamma_{\alpha\eta}\right\} dS\\ &\quad -\frac{1}{2}\int\!\!\int_\Omega \left(\left[\varepsilon_{\alpha\eta}\psi_\beta + \varepsilon_{\beta\eta}\psi_\alpha\right]\chi_{\alpha\beta,\eta} - \varepsilon_{\alpha\eta}\psi_\gamma\gamma_{\alpha,\eta\gamma}\right)\psi_\beta\,d\Omega\\ &= -\frac{1}{2}\int\!\!\int_\Omega \left(\varepsilon_{\alpha\eta}\left[\chi_{\alpha\beta,\eta} + \gamma_{\eta,\alpha\beta}\right] + \varepsilon_{\alpha\eta}\chi_{\beta\alpha,\eta}\right)\psi_\beta\,d\Omega = 0\,.\end{aligned}$$

Eq. (25) follows by considering $-\varepsilon_{\alpha\eta}\gamma_{\alpha,\eta\gamma}\psi_\gamma = \varepsilon_{\alpha\eta}\gamma_{\eta,\alpha\beta}\psi_\beta$. Equation (25) can be referred to as the compatibility condition of the Reissner-Mindlin plate. □

Lemma 2.3. Assume that Ω is a single connected region with boundary Γ, and there is a small single connected region, Ω_e, inside Ω (see Fig. 4). For the function $f(\mathbf{x})$ ($f, f_{,\alpha} \in L^1(\Omega)$), that is discontinuous across the inclusion's boundary Γ_e, the following equation holds:

$$\int\!\!\int_\Omega f_{,\alpha}\,d\Omega = -\oint_{\Gamma_e}[f]n_\alpha\,dS + \oint_\Gamma f n_\alpha\,dS\,, \tag{29}$$

where $[f] := f|_{\mathbf{x}\in\Gamma_e{}^+} - f|_{\mathbf{x}\in\Gamma_e{}^-} = f^+ - f^-$.

Proof: By the Gauss theorem, it is straightforward that

$$\begin{aligned}\int\!\!\int_\Omega f_{,\alpha}\,d\Omega &= \int\!\!\int_{\Omega_e} f_{,\alpha}\,d\Omega + \int\!\!\int_{\Omega/\Omega_e} f_{,\alpha}\,d\Omega\,,\\ &= \oint_{\Gamma_e} f^- n_\alpha\,dS - \oint_{\Gamma_e} f^+ n_\alpha\,dS + \oint_\Gamma f n_\alpha\,dS\\ &= -\oint_{\Gamma_e}[f]n_\alpha\,dS + \oint_\Gamma f n_\alpha\,dS\,.\end{aligned} \tag{30}$$

2.2 Solution of the elliptical inclusion problem

Consider the Green's function of the Reissner-Mindlin plate, which is the solution of the following partial differential equations:

$$m^{G(k)}_{\alpha\beta,\beta} - Q_\alpha{}^{G(k)} + \delta_{\alpha k}\delta(\mathbf{x},\mathbf{x}') = 0\,, \tag{31.1}$$

$$Q^{G(k)}_{\alpha,\alpha} + \delta_{3k}\delta(\mathbf{x},\mathbf{x}') = 0\,, \tag{31.2}$$

where $k = 1,2,3$ and $\alpha,\beta = 1,2$. For the linear isotropic plate, Eqs. (31.1, 2) can be rewritten in terms of deflection and rotations,

$$\begin{aligned}
&D\left[\phi^{G(k)}_{1,11} + \frac{1-\nu}{2}\,\phi^{G(k)}_{1,22} + \frac{1+\nu}{2}\,\phi^{G(k)}_{2,12}\right] - G_p(\phi_1{}^{G(k)} + w^{G(k)}_{,1}) + \delta_{1k}\delta(\mathbf{x},\mathbf{x}') = 0\,,\\
&D\left[\frac{1+\nu}{2}\,\phi^{G(k)}_{1,12} + \frac{1-\nu}{2}\,\phi^{G(k)}_{2,11} + \phi^{G(k)}_{2,22}\right] - G_p(\phi_2{}^{G(k)} + w^{G(k)}_{,2}) + \delta_{2k}\delta(\mathbf{x},\mathbf{x}') = 0\,,\\
&G_p[\phi^{G(k)}_{1,1} + \phi^{G(k)}_{2,2} + \nabla^2 w^{G(k)}] + \delta_{3k}\delta(\mathbf{x},\mathbf{x}') = 0\,,
\end{aligned} \tag{32}$$

where $r := |\mathbf{x}' - \mathbf{x}| = \sqrt{(x_1{}' - x_1)^2 + (x_2{}' - x_2)^2}$, and the derivatives in Eqs. (31) and (32) are taken with respect to the variable $\mathbf{x}'$.

Let

$$\mathbf{U}^{G(k)} := (u_1{}^{G(k)}, u_2{}^{G(k)}, u_3{}^{G(k)}) = (\phi_1{}^{G(k)}, \phi_2{}^{G(k)}, w^{G(k)})\,. \tag{33}$$

The system of Eqs. (32) can be cast into compact form:

$$L_{ij}(\partial\mathbf{x}')\,u_j{}^{G(k)} + \delta_{ik}\delta(\mathbf{x},\mathbf{x}') = 0\,, \tag{34}$$

where

$$L_{\alpha\beta}(\partial\mathbf{x}') := D\,\frac{(1-\nu)}{2}\left[(\nabla^2_{\mathbf{x}'} - \lambda^2)\,\delta_{\alpha\beta} + \frac{1+\nu}{1-\nu}\,\frac{\partial^2}{\partial x_\alpha{}'\partial x_\beta{}'}\right], \tag{35}$$

$$L_{\alpha 3}(\partial\mathbf{x}') := -L_{3\alpha}(\partial\mathbf{x}') = -D\,\frac{(1-\nu)}{2}\,\lambda^2\,\frac{\partial}{\partial x_\alpha{}'}\,, \tag{36}$$

$$L_{33}(\partial\mathbf{x}') := D\,\frac{(1-\nu)}{2}\,\lambda^2\nabla^2_{\mathbf{x}'}\,. \tag{37}$$

The solution of Eq. (34), i.e., the Green's function of Reissner-Mindlin plates, is given by Vander Weeën [41] as follows:

$$\phi_\alpha{}^{G(\zeta)} = \frac{1}{2\pi D}\left\{\frac{2}{1-\nu}\,[B(z)\delta_{\zeta\alpha} - A(z)r_{,\alpha}r_{,\zeta}] - \frac{1}{2}\,\delta_{\alpha\zeta}\left(\ln z - \frac{1}{2}\right) - \frac{1}{2}\,r_{,\alpha}r_{,\zeta}\right\}, \tag{38}$$

$$w^{G(\alpha)} = -\phi_\alpha{}^{G(3)} = \frac{1}{8\pi D}\,(2\ln z - 1)\,rr_{,\alpha}\,, \tag{39}$$

$$w^{G(3)} = -\frac{1}{2\pi D}\,\frac{1}{\lambda^2}\left[\frac{2}{1-\nu}\,\ln z - \frac{1}{4}\,z^2(\ln z - 1)\right], \tag{40}$$

where $z = \lambda r$, and

$$A(z) = K_0(z) + \frac{2}{z}\left[K_1(z) - \frac{1}{z}\right], \tag{41}$$

$$B(z) = K_0(z) + \frac{1}{z}\left[K_1(z) - \frac{1}{z}\right], \tag{42}$$

and $K_0(z), K_1(z)$ are the zero-th order and first-order Macdonald's functions (the second kind modified Bessel functions), respectively.

Considering

$$r_{,\alpha} := \frac{\partial r}{\partial x_\alpha{}'} = \cos\left(r, x_\alpha{}' - x_\alpha\right), \qquad x_\alpha{}' - x_\alpha = r r_{,\alpha},$$

one may derive that

$$\begin{aligned} m_{\alpha\beta}^{G(\zeta)} = \frac{\lambda}{2\pi}\Bigg\{ \Bigg(& B'(z)\left(\delta_{\zeta\alpha} r_{,\beta} + \delta_{\zeta\beta} r_{,\alpha}\right) - 2A'(z)\, r_{,\alpha} r_{,\beta} r_{,\zeta} \\ & - \frac{A(z)}{z}\left(\delta_{\zeta\beta} r_{,\alpha} + \delta_{\zeta\alpha} r_{,\beta} + 2\delta_{\alpha\beta} r_{,\zeta} - 4 r_{,\alpha} r_{,\beta} r_{,\zeta}\right)\Bigg) \\ & - \frac{(1-\nu)}{2z}\left(\delta_{\zeta\alpha} r_{,\beta} + \delta_{\zeta\beta} r_{,\alpha} + \delta_{\alpha\beta} r_{,\zeta} - 2 r_{,\alpha} r_{,\beta} r_{,\zeta}\right) - \nu \frac{r_{,\zeta}}{z}\,\delta_{\alpha\beta}\Bigg\}, \end{aligned} \tag{43}$$

$$m_{\alpha\beta}^{G(3)} = -\frac{(1-\nu)}{8\pi}\left\{\left[2\frac{(1+\nu)}{(1-\nu)}\ln z - 1\right]\delta_{\alpha\beta} + 2 r_{,\alpha} r_{,\beta}\right\}, \tag{44}$$

$$Q_\alpha{}^{G(\zeta)} = \frac{\lambda^2}{2\pi}\left[B(z)\,\delta_{\zeta\alpha} - A(z)\, r_{,\alpha} r_{,\zeta}\right], \tag{45}$$

$$Q_\alpha{}^{G(3)} = -\frac{1}{2\pi}\frac{r_{,\alpha}}{r}. \tag{46}$$

It is found that $m_{\alpha\beta}^{G(\zeta)}, Q_\alpha{}^{G(\zeta)}, Q_\alpha{}^{G(3)} \to 0$ as $r \to \infty$. However, this is not true for $m_{\alpha\beta}^{G(3)}$; nonetheless,

$$m_{\alpha\beta,\gamma}^{G(3)} = -\frac{1}{4\pi r}\left\{(1+\nu)\,\delta_{\alpha\beta} r_{,\gamma} + (1-\nu)\left(\delta_{\alpha\gamma} r_{,\beta} + \delta_{\beta\gamma} r_{,\alpha} - 2 r_{,\alpha} r_{,\beta} r_{,\gamma}\right)\right\}, \tag{47}$$

which indicates that $m_{\alpha\beta,\gamma}^{G(3)} \to 0$ as $r \to \infty$.

Now turn to the corresponding inclusion problem. Consider an infinite plate with no external loads, in which there is an elliptical inclusion embedded at the center of the plate (see Fig. 5). An eigen-curvature/eigen-rotation field is prescribed inside the elliptical inclusion:

$$\chi_{\alpha\beta}^* = \begin{cases} \chi_{\alpha\beta}^*(\mathbf{x}) & \forall\, \mathbf{x} \in \Omega_e \\ 0 & \forall\, \mathbf{x} \in \Omega/\Omega_e \end{cases} \tag{48}$$

$$\gamma_\alpha{}^* = \begin{cases} \gamma_\alpha{}^*(\mathbf{x}) & \forall\, \mathbf{x} \in \Omega_e \\ 0 & \forall\, \mathbf{x} \in \Omega/\Omega_e \end{cases} \tag{49}$$

The equilibrium state of the plate is controlled by the residual moments and residual shear forces that are caused by the misfit

$$m_{\alpha\beta,\beta} - Q_\alpha = -F_\alpha = -\left(m_{\alpha\beta,\beta}^* - Q_\alpha{}^*\right), \tag{50}$$

$$Q_{\alpha,\alpha} = -F_3 = -Q_{\alpha,\alpha}^*, \tag{51}$$

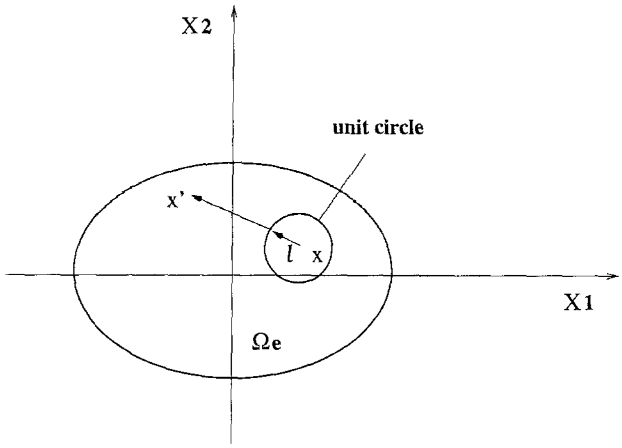


Fig. 5. The integration scheme inside an elliptical inclusion

where $m^*_{\alpha\beta}, Q_\alpha{}^*$ are the eigen-moment and eigen-resultant,

$$m^*_{\alpha\beta} := -D\,\frac{(1-\nu)}{2}\left(\phi^*_{\alpha,\beta} + \phi^*_{\beta,\alpha} + \frac{2\nu}{1-\nu}\,\phi^*_{\gamma,\gamma}\delta_{\alpha\beta}\right), \tag{52}$$

$$Q_\alpha{}^* := -G_p(\phi_\alpha{}^* + w^*_{,\alpha})\,. \tag{53}$$

From Betti's reciprocal theorem, the boundary integral equation for the Reissner-Mindlin plate reads[3] (Vander Weeën [41], Karam and Telles [21]):

$$\begin{aligned}\delta_{ik}u_i(\mathbf{x}) &+ \int_{\Gamma_\infty} m^{G(k)}_{\alpha\beta}(\mathbf{x},\mathbf{x}')\,n_\beta(\mathbf{x}')\,u_\alpha(\mathbf{x}')\,dS' + \int_{\Gamma_\infty} Q_\alpha{}^{G(k)}(\mathbf{x},\mathbf{x}')\,n_\alpha(\mathbf{x}')\,u_3(\mathbf{x}')\,dS' \\ &= \int_{\Gamma_\infty} m_{\alpha\beta}(\mathbf{x}')\,n_\beta(\mathbf{x}')\,u_\alpha{}^{G(k)}(\mathbf{x},\mathbf{x}')\,dS' + \int_{\Gamma_\infty} Q_\alpha(\mathbf{x}')\,n_\alpha(\mathbf{x}')\,u_3{}^{G(k)}(\mathbf{x},\mathbf{x}')\,dS'\,, \\ &\quad - \oint_{\partial\Omega_e} [m_{\alpha\beta}]\,(\mathbf{x}')\,n_\beta(\mathbf{x}')\,u_\alpha{}^{G(k)}(\mathbf{x},\mathbf{x}')\,dS' - \oint_{\partial\Omega_e} [Q_\alpha]\,(\mathbf{x}')\,n_\alpha(\mathbf{x}')\,u_3{}^{G(k)}(\mathbf{x},\mathbf{x}')\,dS' \\ &\quad + \iint_\Omega F_\alpha(\mathbf{x}')\,\phi_\alpha{}^{G(k)}(\mathbf{x},\mathbf{x}')\,d\Omega' + \iint_\Omega F_3(\mathbf{x}')\,u_3{}^{G(k)}(\mathbf{x},\mathbf{x}')\,d\Omega'\,.\end{aligned} \tag{54}$$

It is reasonable to assume that the induced resultant disturbances vanish at infinity (not the induced displacement field in general), i.e.,

$$m_{\alpha\beta}, Q_\alpha \to 0\,, \quad r \to \infty\,, \tag{55}$$

and at the matrix/inclusion interface,

$$[m_{\alpha\beta}] + [m^*_{\alpha\beta}] = 0\,, \qquad \forall\,\mathbf{x}' \in \partial\Omega_e\,, \tag{56}$$

$$[Q_\alpha] + [Q_\alpha{}^*] = 0\,, \qquad \forall\,\mathbf{x}' \in \partial\Omega_e\,. \tag{57}$$

It can then be readily shown that

$$\begin{aligned}\phi_\zeta(\mathbf{x}) &= \iint_\Omega (m^*_{\alpha\beta,\beta} - Q_\alpha{}^*)\,(\mathbf{x}')\,\phi_\alpha{}^{G(\zeta)}(\mathbf{x},\mathbf{x}')\,d\Omega + \oint_{\partial\Omega_e} [m^*_{\alpha\beta}]\,(\mathbf{x}')\,\phi_\alpha{}^{G(\zeta)}(\mathbf{x},\mathbf{x}')\,n_\beta(\mathbf{x}')\,dS'\,, \\ &\quad + \iint_\Omega Q^*_{\alpha,\alpha}(\mathbf{x}')\,w^{G(\zeta)}(\mathbf{x},\mathbf{x}')\,d\Omega' + \oint_{\partial\Omega_e} [Q_\alpha{}^*]\,w^{G(\zeta)}(\mathbf{x},\mathbf{x}')\,n_\alpha(\mathbf{x}')\,dS' \\ &= -\iint_{\Omega_e} \left(m^*_{\alpha\beta}(\mathbf{x}')\,\chi^{G(\zeta)}_{\alpha\beta}(\mathbf{x},\mathbf{x}') + Q_\alpha{}^*(\mathbf{x}')\,\gamma_\alpha{}^{G(\zeta)}(\mathbf{x},\mathbf{x}')\right) d\Omega'\end{aligned} \tag{58}$$

[3] For smooth boundary and zero vertical external load only.

and

$$\begin{aligned} w(\mathbf{x}) &= \iint_{\Omega} (m^*_{\alpha\beta,\beta} - Q_\alpha{}^*)(\mathbf{x}')\, \phi_\alpha{}^{G(3)}(\mathbf{x},\mathbf{x}')\, d\Omega' + \oint_{\partial\Omega_e} [m^*_{\alpha\beta}](\mathbf{x}')\, \phi_\alpha{}^{G(3)}(\mathbf{x},\mathbf{x}')\, n_\beta(\mathbf{x}')\, dS' \\ &\quad + \iint_{\Omega} Q^*_{\alpha,\alpha}(\mathbf{x}')\, w^{G(3)}(\mathbf{x},\mathbf{x}')\, d\Omega' + \oint_{\partial\Omega_e} [Q_\alpha{}^*](\mathbf{x}')\, w^{G(3)}(\mathbf{x},\mathbf{x}')\, n_\alpha(\mathbf{x}')\, dS' \\ &\quad - \int_{\Gamma_\infty} m^{G(3)}_{\alpha\beta}(\mathbf{x},\mathbf{x}')\, n_\beta(\mathbf{x}')\, \phi_\alpha(\mathbf{x}')\, dS' \\ &= -\iint_{\Omega_e} \left(m^*_{\alpha\beta}(\mathbf{x}')\, \chi^{G(3)}_{\alpha\beta}(\mathbf{x},\mathbf{x}') + Q_\alpha{}^*(\mathbf{x}')\, \gamma_\alpha{}^{G(3)}(\mathbf{x},\mathbf{x}') \right) d\Omega' \\ &\quad - \int_{\Gamma_\infty} m^{G(3)}_{\alpha\beta}(\mathbf{x},\mathbf{x}')\, n_\beta(\mathbf{x}')\, \phi_\alpha(\mathbf{x}')\, dS' . \end{aligned} \tag{59}$$

Note that Lemma (2.3) is used throughout the manipulation.

Since

$$\frac{\partial}{\partial x_\gamma} \int_{\Gamma_\infty} m^{G(3)}_{\alpha\beta}(\mathbf{x},\mathbf{x}')\, n_\beta(\mathbf{x}')\, \phi_\alpha(\mathbf{x}')\, dS' = -\int_{\Gamma_\infty} m^{G(3)}_{\alpha\beta,\gamma}(\mathbf{x},\mathbf{x}')\, n_\beta(\mathbf{x}')\, \phi_\alpha(\mathbf{x}')\, dS' \to 0 \tag{60}$$

the difference between (59) and the expression

$$w(\mathbf{x}) = -\iint_{\Omega_e} \left(m^*_{\alpha\beta}(\mathbf{x}')\, \chi^{G(3)}_{\alpha\beta}(\mathbf{x},\mathbf{x}') + Q_\alpha{}^*(\mathbf{x}')\, \gamma_\alpha{}^{G(3)}(\mathbf{x},\mathbf{x}') \right) d\Omega' \tag{61}$$

is a constant. This also implies that although the displacement may be unbounded for an infinite domain, the associated flexural curvature field and shear deformation field are still finite.

It would be expedient to decompose the induced deformation field into two parts:

$$\phi_\zeta(\mathbf{x}) = \phi_\zeta{}^M(\mathbf{x}) + \phi_\zeta{}^Q(\mathbf{x}) , \tag{62}$$

$$w(\mathbf{x}) = w^M(\mathbf{x}) + w^Q(\mathbf{x}) , \tag{63}$$

where the superscript, M, denotes the deformation field caused by the eigenmoment, and the superscript, Q, denotes the deformation field induced by the eigen-shear-resultant. Utilizing (52)–(53), one can write

$$\phi_\zeta{}^M(\mathbf{x}) := -\iint_{\Omega_e} m^*_{\alpha\beta}(\mathbf{x}')\, \chi^{G(\zeta)}_{\alpha\beta}(\mathbf{x}'-\mathbf{x})\, d\Omega' = \iint_{\Omega_e} \chi^*_{\alpha\beta}(\mathbf{x}')\, m^{G(\zeta)}_{\alpha\beta}(\mathbf{x}'-\mathbf{x})\, d\Omega' , \tag{64}$$

$$\phi_\zeta{}^Q(\mathbf{x}) := -\iint_{\Omega_e} Q_\alpha{}^*(\mathbf{x}')\, \gamma_\alpha{}^{G(\zeta)}(\mathbf{x}'-\mathbf{x})\, d\Omega' = \iint_{\Omega_e} \gamma_\alpha{}^*(\mathbf{x}')\, Q_\alpha{}^{G(\zeta)}(\mathbf{x}'-\mathbf{x})\, d\Omega' , \tag{65}$$

$$w^M(\mathbf{x}) := -\iint_{\Omega_e} m^*_{\alpha\beta}(\mathbf{x}')\, \chi^{G(3)}_{\alpha\beta}(\mathbf{x}'-\mathbf{x})\, d\Omega' = \iint_{\Omega_e} \chi^*_{\alpha\beta}(\mathbf{x}')\, m^{G(3)}_{\alpha\beta}(\mathbf{x}'-\mathbf{x})\, d\Omega' , \tag{66}$$

$$w^Q(\mathbf{x}) = -\iint_{\Omega_e} Q_\alpha{}^*(\mathbf{x}')\, \gamma_\alpha{}^{G(3)}(\mathbf{x}'-\mathbf{x})\, d\Omega' = \iint_{\Omega_e} \gamma_\alpha{}^*(\mathbf{x}')\, Q_\alpha{}^{G(3)}(\mathbf{x}'-\mathbf{x})\, d\Omega' , \tag{67}$$

where $m^{G(\zeta)}_{\alpha\beta}, m^{G(3)}_{\alpha\beta}, Q_\alpha{}^{G(\zeta)}$ and $Q_\alpha{}^{G(3)}$ are given in Eqs. (43)–(46).

To proceed further, one may need the following expressions:

$$\chi^M_{\zeta\eta}(\mathbf{x}) := \frac{1}{2}\left(\phi^M_{\eta,\zeta}(\mathbf{x}) + \phi^M_{\zeta,\eta}(\mathbf{x})\right) = -\frac{1}{2}\iint_{\Omega_e} \chi^*_{\alpha\beta}(\mathbf{x}')\left(m^{G(\eta)}_{\alpha\beta,\zeta}(\mathbf{x}'-\mathbf{x}) + m^{G(\zeta)}_{\alpha\beta,\eta}(\mathbf{x}'-\mathbf{x})\right) d\Omega' , \tag{68}$$

$$\chi^Q_{\zeta\eta}(\mathbf{x}) := \frac{1}{2}\left(\phi^Q_{\eta,\zeta}(\mathbf{x}) + \phi^Q_{\zeta,\eta}(\mathbf{x})\right) = -\frac{1}{2}\iint_{\Omega_e} \gamma_\alpha{}^*(\mathbf{x}')\left(Q^{G(\eta)}_{\alpha,\zeta}(\mathbf{x}'-\mathbf{x}) + Q^{G(\zeta)}_{\alpha,\eta}(\mathbf{x}'-\mathbf{x})\right) d\Omega' , \tag{69}$$

$$\gamma_\zeta{}^M(\mathbf{x}) := \phi_\zeta{}^M(\mathbf{x}) + w^M_{,\eta}(\mathbf{x}) = \iint_{\Omega_e} \chi^*_{\alpha\beta}(\mathbf{x}')\big(m^{G(\zeta)}_{\alpha\beta}(\mathbf{x}'-\mathbf{x}) - m^{G(3)}_{\alpha\beta,\zeta}(\mathbf{x}'-\mathbf{x})\big)\, d\Omega' , \tag{70}$$

$$\gamma_\zeta{}^Q(\mathbf{x}) := \phi_\zeta{}^Q(\mathbf{x}) + w^Q_{,\zeta}(\mathbf{x}) = \iint_{\Omega_e} \gamma_\alpha{}^*(\mathbf{x}')\big(Q_\alpha{}^{G(\zeta)}(\mathbf{x}'-\mathbf{x}) - Q^{G(3)}_{\alpha,\zeta}(\mathbf{x}'-\mathbf{x})\big)\, d\Omega' . \tag{71}$$

Note that the derivatives in the integrands of (68)–(71) are all made with respect to $\mathbf{x}'$.

If the prescribed eigen-curvature and eigen-rotation are uniform inside the elliptical inclusion, Eqs. (68)–(71) can be cast into succinct forms,

$$\chi^M_{\zeta\eta}(\mathbf{x}) = S^M_{\zeta\eta\alpha\beta}(\mathbf{x})\, \chi^*_{\alpha\beta} , \tag{72}$$

$$\chi^Q_{\zeta\eta}(\mathbf{x}) = S^Q_{\zeta\eta\alpha}(\mathbf{x})\, \gamma_\alpha{}^* , \tag{73}$$

$$\gamma_\zeta{}^M(\mathbf{x}) = T^M_{\zeta\alpha\beta}(\mathbf{x})\, \chi^*_{\alpha\beta} , \tag{74}$$

$$\gamma_\zeta{}^Q(\mathbf{x}) = T^Q_{\zeta\alpha}(\mathbf{x})\, \gamma_\alpha{}^* , \tag{75}$$

where

$$S^M_{\zeta\eta\alpha\beta}(\mathbf{x}) = -\frac{1}{2}\iint_{\Omega_e} \big(m^{G(\eta)}_{\alpha\beta,\zeta}(\mathbf{x}'-\mathbf{x}) + m^{G(\zeta)}_{\alpha\beta,\eta}(\mathbf{x}'-\mathbf{x})\big)\, d\Omega' , \tag{76}$$

$$S^Q_{\zeta\eta\alpha}(\mathbf{x}) = -\frac{1}{2}\iint_{\Omega_e} \big(Q^{G(\eta)}_{\alpha,\zeta}(\mathbf{x}'-\mathbf{x}) + Q^{G(\zeta)}_{\alpha,\eta}(\mathbf{x}'-\mathbf{x})\big)\, d\Omega' , \tag{77}$$

$$T^M_{\zeta\alpha\beta}(\mathbf{x}) = \iint_{\Omega_e} \big(m^{G(\zeta)}_{\alpha\beta}(\mathbf{x}'-\mathbf{x}) - m^{G(3)}_{\alpha\beta,\zeta}(\mathbf{x}'-\mathbf{x})\big)\, d\Omega' , \tag{78}$$

$$T^Q_{\zeta\alpha}(\mathbf{x}) = \iint_{\Omega_e} \big(Q_\alpha{}^{G(\zeta)}(\mathbf{x}'-\mathbf{x}) - Q^{G(3)}_{\alpha,\zeta}(\mathbf{x}'-\mathbf{x})\big)\, d\Omega' \tag{79}$$

are the generalized Eshelby tensors in Reissner-Mindlin plate theory.

In general, for elliptical inclusions, $S^M_{\zeta\eta\alpha\beta}, S^Q_{\zeta\eta\alpha}, T^M_{\zeta\alpha\beta}$ and $T^Q_{\zeta\alpha}$ are not constant tensors, even if $\mathbf{x} \in \Omega_e$; consequently they depend on the size of the inclusion. Nevertheless, when the size of the inclusion is sufficiently small, the constant part and the linear parts of the Eshelby tensors should dominate. In what follows, by using asymptotic expansion, we calculate the approximated value of $S^M_{\zeta\eta\alpha\beta}, S^Q_{\zeta\eta\alpha}, T^M_{\zeta\alpha\beta}$ and $T^Q_{\zeta\alpha}$ for $\mathbf{x} \in \Omega_e$ under the restrictions

$$z = \lambda r < 1 , \qquad \text{where} \qquad \lambda := \frac{\sqrt{12}\varkappa(\nu)}{h} . \tag{80}$$

When $|z| < 1$, from Watson [47]

$$K_n(z) = \frac{1}{2}\sum_{k=0}^{n-1} \frac{(-1)^k (n-k-1)!}{k!}\left(\frac{z}{2}\right)^{2k-n} + \frac{(-1)^{n-1}}{2}\sum_{k=0}^{\infty} \frac{(z/2)^{2k+n}}{k!(k+n)!}\left[2\log\frac{z}{2} - \psi(k+1) - \psi(k+n+1)\right] , \tag{81}$$

where $\psi(m+1) = -\gamma + 1 + \frac{1}{2} + \cdots\cdots + \frac{1}{m}$, and $\gamma = 0.577\,215\,66\cdots$ is Euler's constant.

Considering the first order approximation, we take

$$K_0(z) = -\frac{1}{2}\left(2\log\frac{z}{2} - 2\psi(1)\right) + \mathcal{O}(z^2)\,, \tag{82}$$

$$K_1(z) = \frac{z}{4}\left[2\log\frac{z}{2} - \left(\psi(1)+\psi(2)\right)\right] + \frac{1}{2} + \mathcal{O}(z^3)\,. \tag{83}$$

Subsequently,

$$A(z) = -\frac{1}{2}[\psi(1)+\psi(2)] + \mathcal{O}(z^2) = -\frac{1}{2} + \mathcal{O}(z^2)\,, \tag{84}$$

$$B(z) = -\frac{1}{2}\log\left(\frac{z}{2}\right) + \frac{1}{4}\,[3\psi(1) - \psi(2)] + \mathcal{O}(z^2)\,. \tag{85}$$

Substituting the expressions (84) and (85) into (43) and (44) yields

$$m_{\alpha\beta}^{G(\zeta)} = -\frac{1}{4\pi r}\left\{(1-\nu)\left(\delta_{\alpha\zeta}r_{,\beta} + \delta_{\beta\zeta}r_{,\alpha} - \delta_{\alpha\beta}r_{,\zeta}\right) + 2(1+\nu)\,r_{,\alpha}r_{,\beta}r_{,\zeta}\right\} \tag{86}$$

and

$$Q_\alpha{}^{G(\zeta)} = -\frac{\lambda^2}{4\pi}\left\{\left[\log\left(\frac{z}{2}\right) + \frac{2\gamma+1}{2}\right]\delta_{\zeta\alpha} - r_{,\zeta}r_{,\alpha}\right\}. \tag{87}$$

Now, we are in a position to derive the approximated Eshelby tensors for Reissner-Mindlin plates.

2.2.1 $S^{FM}_{\zeta\eta\alpha\beta}$

Assume that the eigen-curvature field, $\chi^*_{\alpha\beta}$, inside the elliptical inclusion is uniform. By adopting the integration scheme shown in Fig. 5 and choosing the polar coordinate system $d\Omega' = r\,dr\,d\theta$, Eq. (64) takes the form:

$$\phi_\zeta{}^M(\mathbf{x}) = -\frac{\chi^*_{\alpha\beta}}{4\pi}\int_0^{2\pi}\int_0^{\varrho}[(1-\nu)(\delta_{\alpha\zeta}\ell_\beta + \delta_{\beta\zeta}\ell_\alpha - \delta_{\alpha\beta}\ell_\zeta) + 2(1+\nu)\,\ell_\alpha\ell_\beta\ell_\zeta]\,dr\,d\theta\,, \tag{88}$$

where $\ell_\zeta := r_{,\zeta} = (x_\zeta' - x_\zeta)/r$, and $\varrho = \varrho(\ell_\mu, x_\mu)$ is the root of the quadratic equation

$$\frac{(x_1+\varrho\ell_1)^2}{{a_1}^2} + \frac{(x_2+\varrho\ell_2)^2}{{a_2}^2} = 1\,. \tag{89}$$

Solving Eq. (89), we obtain

$$\varrho(\ell_\mu, x_\mu) = -\frac{f}{g} \pm \sqrt{\frac{f^2}{g^2} + \frac{e}{g}}, \tag{90}$$

where

$$f := \lambda_\mu x_\mu\,, \tag{91}$$

$$\lambda_\mu := \frac{\ell_\mu}{{a_\mu}^2}\,, \tag{92}$$

$$g := \frac{\ell_1{}^2}{a_1{}^2} + \frac{\ell_2{}^2}{a_2{}^2}\,, \tag{93}$$

$$e := 1 - \left(\frac{x_1{}^2}{a_1{}^2} + \frac{x_2{}^2}{a_2{}^2}\right). \tag{94}$$

Because $\pm\sqrt{f^2/g^2 + e/g}$ is an even function of $\ell(\ell_1, \ell_2)$, subsequently,

$$\phi_\zeta{}^M(\mathbf{x}) = \frac{\chi^*_{\alpha\beta}}{4\pi} \int_0^{2\pi} \left(\frac{\lambda_\mu x_\mu}{g}\right) \{(1-\nu)[\delta_{\alpha\zeta}\ell_\beta + \delta_{\beta\zeta}\ell_\alpha - \delta_\alpha\beta\ell_\zeta] + 2(1+\nu)\,\ell_\alpha\ell_\beta\ell_\zeta\}\, d\theta\,, \tag{95}$$

and furthermore

$$\phi^M_{\zeta,\eta} = \frac{\chi^*_{\alpha\beta}}{4\pi} \int_0^{2\pi} \left(\frac{\lambda_\mu \delta_{\mu\eta}}{g}\right) g_{\zeta\alpha\beta}\, d\theta = \frac{\chi^*_{\alpha\beta}}{4\pi} \int_0^{2\pi} \frac{\lambda_\eta g_{\zeta\alpha\beta}}{g}\, d\theta\,, \tag{96}$$

where $g_{\zeta\alpha\beta} := (1-\nu)[\delta_{\alpha\zeta}\ell_\beta + \delta_{\beta\zeta}\ell_\alpha - \delta_{\alpha\beta}\ell_\zeta] + 2(1+\nu)\,\ell_\alpha\ell_\beta\ell_\zeta$.

Symmetrizing the integral representation (96), we have

$$\chi_{\zeta\eta} = \frac{1}{2}\,(\phi^M_{\zeta,\eta} + \phi^M_{\eta,\zeta}) = \frac{\chi^*_{\alpha\beta}}{8\pi} \int_0^{2\pi} \frac{(\lambda_\eta g_{\zeta\alpha\beta} + \lambda_\zeta g_{\eta\alpha\beta})}{g}\, d\theta\,. \tag{97}$$

Define tensor $S^{FM}_{\zeta\eta\alpha\beta}$ as

$$S^{FM}_{\zeta\eta\alpha\beta} := \frac{1}{8\pi} \int_0^{2\pi} \frac{(\lambda_\zeta g_{\eta\alpha\beta} + \lambda_\eta g_{\zeta\alpha\beta})}{g}\, d\theta\,. \tag{98}$$

We then obtain the desired result,

$$\chi_{\zeta\eta} = S^{FM}_{\zeta\eta\alpha\beta}\chi^*_{\alpha\beta}\,. \tag{99}$$

Remark 2.2.1. The first superscript letter, F, stands for the first-order approximation. Unlike the tensor $S^M_{\zeta\eta\alpha\beta}$, its first-order approximation $S^{FM}_{\zeta\eta\alpha\beta}$ is a constant tensor, and independent from the size of the inclusion. In Appendix A, a detailed list is given for every component of $S^{FM}_{\zeta\eta\alpha\beta}$. 2. The existence of integrals (68), and (71) can be shown in a similar fashion as done by Kellog [24], and Torquato [40]. □

2.2.2 $S^{FQ}_{\zeta\eta\alpha\beta}$

Imposing a uniform eigen-shear-deformation, $\gamma_\alpha{}^*$, inside the elliptical inclusion, one may obtain the induced rotation field by virtue of (65) and (87),

$$\phi_\zeta{}^Q(\mathbf{x}) = -\frac{\lambda^2}{4\pi} \iint_{\Omega_e} \gamma_\alpha{}^* \left\{ \left[\log\left(\frac{z}{2}\right) + \frac{1+2\gamma}{2}\right] \delta_{\zeta\alpha} - r_{,\zeta} r_{,\alpha} \right\} d\Omega\,. \tag{100}$$

After simple manipulation, one can find that

$$\chi^Q_{\zeta\eta} = \frac{\lambda^2 \gamma_\alpha{}^*}{4\pi} \int_0^{2\pi} \left(\frac{\lambda_\mu x_\mu}{g}\right) (\delta_{\zeta\eta} r_{,\alpha} - 2 r_{,\zeta} r_{,\eta} r_{,\alpha})\, d\theta\,, \tag{101}$$

and Eq. (101) becomes

$$\chi^Q_{\zeta\eta} = S^{FQ}_{\zeta\eta\alpha\beta}\gamma_\alpha{}^* x_\beta \,, \tag{102}$$

where

$$S^{FQ}_{\zeta\eta\alpha\beta} := \frac{\lambda^2}{4\pi}\int\limits_0^{2\pi} \frac{(\delta_{\zeta\eta}\ell_\alpha\ell_\beta - 2\ell_\zeta\ell_\eta\ell_\alpha\ell_\beta)}{a_\beta{}^2 g}\, d\theta \,. \tag{103}$$

Remark 2.3. For a circular inclusion, one may find that

$$S^{FQ}_{\zeta\eta\alpha\beta} = \frac{\lambda^2}{8}\,(\delta_{\zeta\eta}\delta_{\alpha\beta} - \delta_{\zeta\alpha}\delta_{\eta\beta} - \delta_{\eta\alpha}\delta_{\zeta\beta}) \,. \tag{104}$$

Thus,

$$\chi^Q_{\zeta\eta} = \frac{\lambda^2}{8}\,(\delta_{\zeta\eta}\gamma_\alpha{}^* x_\alpha - \gamma_\zeta{}^* x_\eta - \gamma_\eta{}^* x_\zeta) \,. \tag{105}$$

Obviously,

$$< \chi^Q_{\zeta\eta} > = \frac{1}{\Omega_e}\iint\limits_{\Omega_e} \chi^Q_{\zeta\eta}\, d\Omega = 0 \,, \tag{106}$$

because $\iint\limits_{\Omega_e} x_\mu \, d\Omega = 0$ for elliptical inclusions. □

2.2.3 $T^{FM}_{\zeta\alpha\beta\mu}$

By virtue of (70), (43), and (44), one may find that

$$\begin{aligned}
\gamma_\zeta{}^M &= \iint\limits_{\Omega_e} \chi^*_{\alpha\beta}(m^{G(\zeta)}_{\alpha\beta} - m^{G(3)}_{\alpha\beta,\zeta})\, d\Omega \\
&= \frac{\chi^*_{\alpha\beta} x_\mu}{4\pi}\int\limits_0^{2\pi} \Bigg\{ \left(\frac{\ell_\mu}{a_\mu{}^2 g}\right) [(1-\nu)\,(\delta_{\alpha\zeta}\ell_\beta + \delta_{\beta\zeta}\ell_\alpha + \delta_{\alpha\beta}\ell_\zeta) + 2(1+\nu)\,\ell_\alpha\ell_\beta\ell_\zeta] \\
&\qquad - [(1+\nu)\,\delta_{\alpha\beta}\ell_\zeta + (1-\nu)\,(\delta_{\alpha\zeta} + \delta_{\beta\zeta}\ell_\alpha - 2\ell_\alpha\ell_\beta\ell_\zeta)] \Bigg\}\, d\theta \\
&= \frac{\chi^*_{\alpha\beta} x_\mu}{4\pi}\int\limits_0^{2\pi} \Bigg\{ \left(\frac{\ell_\mu}{a_\mu{}^2 g}\right) [(1-\nu)\,(\delta_{\alpha\zeta}\ell_\beta + +\delta_{\beta\zeta}\ell_\alpha) - (3-\nu)\,\delta_{\alpha\beta}\ell_\zeta + 4\ell_\alpha\ell_\beta\ell_\zeta] \Bigg\}\, d\theta \,.
\end{aligned} \tag{107}$$

Let

$$T^{FM}_{\zeta\alpha\beta\mu} := \frac{1}{4\pi}\int\limits_0^{2\pi} \Bigg\{ \left(\frac{\ell_\mu}{a_\mu{}^2 g}\right) [(1-\nu)\,(\delta_{\alpha\zeta}\ell_\beta + +\delta_{\beta\zeta}\ell_\alpha) - (3-\nu)\,\delta_{\alpha\beta}\ell_\zeta + 4\ell_\alpha\ell_\beta\ell_\zeta] \Bigg\}\, d\theta \,. \tag{108}$$

Then, Eq. (107) becomes

$$\gamma_\zeta{}^M = T^{FM}_{\zeta\alpha\beta\mu}\chi^*_{\alpha\beta} x_\mu \,. \tag{109}$$

For a circular inclusion, it is not difficult to find that

$$T^{FM}_{\zeta\alpha\beta\mu} = \frac{(2-\nu)}{4}\left(\delta_{\alpha\zeta}\delta_{\beta\mu} + \delta_{\beta\zeta}\delta_{\alpha\mu} - \delta_{\alpha\beta}\delta_{\zeta\mu}\right), \tag{110}$$

and therefore

$$\gamma_\zeta{}^M = \frac{(2-\nu)}{4}\,\chi^*_{\zeta\mu}x_\mu + \chi^*_{\mu\zeta}x_\mu - \chi^*_{\alpha\alpha}x_\zeta \tag{111}$$

which also leads to

$$<\gamma_\zeta{}^M> := \frac{1}{\Omega_e}\iint_{\Omega_e} \gamma_\zeta{}^M\, d\Omega = 0\,. \tag{112}$$

2.2.4 $T^{FQ}_{\alpha\beta}$

Substituting (87) into the first term in the right-hand side of (71) yields

$$\begin{aligned}\iint_{\Omega_e} \gamma_\beta{}^* Q_\beta{}^{G(\alpha)}\, d\Omega &= -\frac{\lambda^2\gamma_\alpha{}^*}{4\pi}\iint_{\Omega_e}\left\{\left[\log\left(\frac{z}{2}\right) + \frac{2\gamma+1}{2}\right]\delta_{\alpha\beta} - r_{,\alpha}r_{,\beta}\right\} r\,dr\,d\theta \\ &= -\frac{\gamma_\alpha{}^*}{8\pi}\int_0^{2\pi}\left\{z^2\left(\left[\log\left(\frac{z}{2}\right)+\gamma\right]\delta_{\alpha\beta} - \ell_\alpha\ell_\beta\right)\right\}_{z=\lambda\varrho} d\theta\,, \\ &\sim \mathcal{O}(z^2)\end{aligned} \tag{113}$$

where $\varrho \leq \operatorname{diam}\{\Omega_e\} \leq \lambda^{-1}$. Thus, as the first-order approximation, the first term in the right-hand side of (71) can be neglected. In consequence,

$$\gamma_\alpha{}^Q = -\iint_{\Omega_e} \gamma_\beta{}^* Q^{G(3)}_{\beta,\alpha}\, d\Omega = \frac{\gamma_\beta{}^*}{2\pi}\int_0^{2\pi}\left(\frac{\lambda_\alpha\ell_\beta}{g}\right) d\theta = \frac{\gamma_\beta{}^*}{2\pi}\int_0^{2\pi}\frac{\ell_\alpha\ell_\beta}{a_\alpha{}^2 g}\, d\theta\,. \tag{114}$$

Let

$$T^{FQ}_{\alpha\beta} = \frac{1}{2\pi}\int_0^{2\pi}\frac{\ell_\alpha\ell_\beta}{a_\alpha{}^2 g}\, d\theta\,. \tag{115}$$

We have the result

$$\gamma_\alpha{}^Q = T^{FQ}_{\alpha\beta}\gamma_\beta{}^*\,. \tag{116}$$

3 Variational inequalities for Reissner-Mindlin plates

3.1 Some averaging properties of thick plates

To facilitate the presentation, a few definitions are in order. Denote the overall average of any function, $f \in L^1(\Omega)$, as

$$\bar{f} := <f> := \frac{1}{|\Omega|}\iint_\Omega f\, d\Omega \tag{117}$$

and Sym^o as the space of symmetric second-order tensors in $\mathbb{R}^2$ in which all the elements hold constant values.

For convenience, we define the following function spaces:

1. for $w \in C^2(\Omega)$ and $\phi_\alpha \in C^1(\Omega)$,

$$\mathcal{W}_1(\Omega) := \{w \,|\, w = \hat{w}\,, \quad \forall \mathbf{x} \in S_u\}\,, \tag{118}$$

$$\mathcal{W}_2(\Omega) := \left\{ w \,\middle|\, w = <\gamma_\alpha> x_\alpha - \frac{1}{2} <\chi_{\alpha\beta}> x_\alpha x_\beta\,, \quad \forall \mathbf{x} \in S_u \right\}, \tag{119}$$

$$\Phi_1(\Omega) := \{\phi_\alpha \,|\, \phi_n = \hat{\phi}_n \quad \& \quad \phi_s = \hat{\phi}_s\,, \quad \forall \mathbf{x} \in S_u\}\,, \tag{120}$$

$$\Phi_2(\Omega) := \{\phi_\alpha \,|\, \phi_\alpha = <\chi_{\alpha\beta}> x_\beta\,, \quad \forall \mathbf{x} \in S_u\}\,, \tag{121}$$

and

$$\mathcal{D}_1(\Omega) = \{(\phi_\alpha, w) \,|\, \phi_\alpha \in \Phi_1(\Omega)\,, \quad w \in \mathcal{W}_1(\Omega)\}\,, \tag{122}$$

$$\mathcal{D}_2(\Omega) = \{(\phi_\alpha, w) \,|\, \phi_\alpha \in \Phi_2(\Omega)\,, \quad w \in \mathcal{W}_2(\Omega)\}\,, \tag{123}$$

$$\mathcal{D}_3(\Omega) = \big\{\{u_i\} = (\phi_\alpha, w) \,|\, L_{ij} u_j = 0\,, \quad (\phi_\alpha, w) \in \mathcal{D}_1(\Omega)\big\}\,. \tag{124}$$

2. for $\chi_{\alpha,\beta}, \gamma_\alpha \in C^1(\Omega)$,

$$\mathcal{E}_0(\Omega) = \left\{ (\chi_{\alpha\beta}, \gamma_\alpha) \,\middle|\, \chi_{\alpha\beta} = \frac{1}{2}\left(\phi_{\alpha,\beta} + \phi_{\beta,\alpha}\right), \quad \gamma_\alpha = \phi_\alpha + w_{,\alpha}; \right\}, \tag{125}$$

$$\mathcal{E}_1(\Omega) = \{(\chi_{\alpha\beta}, \gamma_\alpha) \in \mathcal{E}_0(\Omega) \,|\, (\phi_\alpha, w) \in \mathcal{D}_1(\Omega)\}\,, \tag{126}$$

$$\mathcal{E}_2(\Omega) = \{(\chi_{\alpha\beta}, \gamma_\alpha) \in \mathcal{E}_0(\Omega) \,|\, (\phi_\alpha, w) \in \mathcal{D}_2(\Omega)\}\,, \tag{127}$$

$$\mathcal{E}_3(\Omega) = \{(\chi_{\alpha\beta}, \gamma_\alpha) \in \mathcal{E}_0(\Omega) \,|\, (\phi_\alpha, w) \in \mathcal{D}_3(\Omega)\}\,. \tag{128}$$

3. for $m_{\alpha\beta}, Q_\alpha \in C^1(\Omega)$,

$$\mathcal{S}_0(\Omega) := \{(m_{\alpha\beta}, Q_\alpha) \,|\, m_{\alpha\beta,\beta} + Q_\alpha = 0\,, \quad Q_{\alpha,\alpha} = 0; \}\,, \tag{129}$$

$$\begin{aligned}\mathcal{S}_1(\Omega) := \{&(m_{\alpha\beta}, Q_\alpha) \,|\, m_{\alpha\beta,\beta} + Q_\alpha = 0\,, \quad Q_{\alpha,\alpha} = 0;\\ &M_{ns} = \hat{M}_{ns}\,, \quad M_n = \hat{M}_n\,, \quad Q_n = \hat{Q}_n\,, \quad \forall \mathbf{x} \in S_F\}\,,\end{aligned} \tag{130}$$

$$\begin{aligned}\mathcal{S}_2(\Omega) := \{&(m_{\alpha\beta}, Q_\alpha) \,|\, m_{\alpha\beta} = <m_{\alpha\beta}> + \sigma^0_{\alpha\beta} \quad \& \quad Q_\alpha = <Q_\alpha>\,, \quad \text{and}\\ &\sigma^0_{\zeta\alpha} x_\beta + \sigma^0_{\zeta\beta} x_\alpha = <Q_\zeta> x_\alpha x_\beta \quad \text{and} \quad Q_\alpha = <Q_\alpha>\,, \quad \forall \mathbf{x} \in S_F\}\,.\end{aligned} \tag{131}$$

A basic task of micromechanics is to establish the relationships between macroscopic variables (at the level of meso-area-element), or averaging variables, because in general the relationships between microscopic variables do not carry through into the mesoscopic level, unless certain provisions are mandated. On the other hand, due to the unique mathematical structure of its governing equations, the Reissner-Mindlin plate posts specific restrictions on boundary conditions such that the relationships between micro-variables can be extended to mesoscopic variables.

Lemma 3.1. Suppose $\phi_\alpha = \chi^0_{\alpha\beta} x_\beta \quad \forall \mathbf{x} \in \partial\Omega$. Then,

$$<\chi_{\alpha\beta}> = \chi^0_{\alpha\beta}\,. \tag{132}$$

Proof: By definition, it is straightforward that

$$< \chi_{\alpha\beta} > = \frac{1}{|\Omega|} \iint\limits_{\Omega} \frac{1}{2} (\phi_{\alpha,\beta} + \phi_{\beta,\alpha}) \, d\Omega \,,$$
$$= \frac{1}{2|\Omega|} \oint\limits_{\partial\Omega} (\phi_\alpha n_\beta + \phi_\beta n_\alpha) \, dS$$
$$= \frac{1}{2|\Omega|} \oint\limits_{\partial\Omega} (\chi^0_{\alpha\gamma} x_\gamma n_\beta + \chi^0_{\beta\gamma} x_\gamma n_\alpha) \, dS$$
$$= \frac{1}{2|\Omega|} \iint\limits_{\Omega} (\chi^0_{\alpha\gamma} \delta_{\gamma\beta} + \chi^0_{\beta\gamma} \delta_{\gamma\alpha}) \, d\Omega = \chi^0_{\alpha\beta} \,. \qquad \square \quad (133)$$

Corollary 3.1. If $\phi_\alpha = 0 \,, \quad \forall \mathbf{x} \in \partial\Omega$ then

$$< \chi_{\alpha\beta} > = 0 \,. \qquad (134)$$

The proof is trivial.

Lemma 3.2. Suppose $w = ({\gamma_\alpha}^0 - < \phi_\alpha >) \, x_\alpha \quad \forall \mathbf{x} \in \partial\Omega$. Then,

$$< \gamma_\alpha > = {\gamma_\alpha}^0 \,. \qquad (135)$$

Proof: From the definition

$$< \gamma_\alpha > = \frac{1}{|\Omega|} \iint\limits_{\Omega} (\phi_\alpha + w_{,\alpha}) \, d\Omega$$
$$= < \phi_\alpha > + \frac{1}{|\Omega|} \oint\limits_{\partial\Omega} w n_\alpha \, dS$$
$$= < \phi_\alpha > + \frac{1}{|\Omega|} \oint\limits_{\partial\Omega} ({\gamma_\beta}^0 - < \phi_\beta >) \, x_\beta n_\alpha \, dS$$
$$= < \phi_\alpha > + \frac{1}{|\Omega|} \iint\limits_{\Omega} ({\gamma_\beta}^0 - < \phi_\beta >) \, \delta_{\alpha\beta} \, d\Omega = {\gamma_\alpha}^0 \,. \qquad \square \quad (136)$$

Lemma 3.3. (i) Suppose $Q_{\alpha,\alpha} = 0$ and $Q_\alpha = {Q_\alpha}^0 = \text{const.} \quad \forall \mathbf{x} \in \partial\Omega$. Then,

$$< Q_\alpha > = {Q_\alpha}^0 \,. \qquad (137)$$

(ii) Suppose $Q_\alpha = m_{\alpha\beta,\beta}$ and $m_{\alpha\beta} = m^0_{\alpha\beta} = \text{const.} \quad \forall \mathbf{x} \in \partial\Omega$. Then,

$$< Q_\alpha > = 0 \,. \qquad (138)$$

Proof: (i) Since $Q_{\alpha,\alpha} = 0$, then $Q_\alpha = \left(Q_\beta x_\alpha\right)_{,\beta}$, and therefore

$$< Q_\alpha > = \frac{1}{|\Omega|} \iint\limits_{\Omega} Q_\alpha \, d\Omega = \frac{1}{|\Omega|} \iint\limits_{\Omega} \left(Q_\beta x_\alpha\right)_{,\beta} d\Omega$$
$$= \frac{1}{|\Omega|} \oint\limits_{\partial\Omega} Q_\beta x_\alpha n_\beta \, dS = \frac{1}{|\Omega|} \oint\limits_{\partial\Omega} {Q_\beta}^0 x_\alpha n_\beta \, dS$$
$$= \frac{{Q_\beta}^0}{|\Omega|} \iint\limits_{\Omega} \delta_{\alpha\beta} \, d\Omega = {Q_\alpha}^0 \,. \qquad (139)$$

(ii) Since $Q_\alpha = m_{\alpha\beta,\alpha}$,

$$\begin{aligned}<Q_\alpha> &= \frac{1}{|\Omega|}\iint_\Omega Q_\alpha \, d\Omega = \frac{1}{|\Omega|}\iint_\Omega m_{\alpha\beta,\beta}\, d\Omega \\ &= \frac{1}{|\Omega|}\oint_{\partial\Omega} m_{\alpha\beta} n_\beta \, dS = \frac{1}{|\Omega|}\oint_{\partial\Omega} m^0_{\alpha\beta} n_\beta \, dS \\ &= 0 . \end{aligned} \qquad \square \ (140)$$

Lemma 3.4. Suppose $m_{\alpha\beta,\beta} - Q_\alpha = 0, Q_{\alpha,\alpha} = 0$. The following equality holds:

$$\frac{1}{|\Omega|}\iint_\Omega m_{\alpha\beta}\, d\Omega = \frac{1}{2|\Omega|}\oint_{\partial\Omega} \{m_{\zeta\alpha} x_\beta n_\zeta + m_{\eta\beta} x_\alpha n_\eta - m_{\zeta\eta,\eta} x_\alpha x_\beta n_\zeta\}\, dS . \qquad (141)$$

Proof: First, one may verify the following identity:

$$m_{\alpha\beta} = \frac{\partial(m_{\zeta\alpha} x_\beta)}{\partial x_\zeta} + \frac{\partial(m_{\eta\beta} x_\alpha)}{\partial x_\eta} + \frac{1}{2} m_{\zeta\eta,\zeta\eta} x_\alpha x_\beta - \frac{1}{2}\frac{\partial^2(m_{\zeta\eta} x_\alpha x_\beta)}{\partial x_\zeta \partial x_\eta} . \qquad (142)$$

Second,

$$m_{\zeta\eta,\eta} - Q_\zeta = 0 \quad \text{and} \quad Q_{\zeta,\zeta} = 0 \Rightarrow m_{\zeta\eta,\zeta\eta} = 0 ,$$

hence

$$m_{\alpha\beta} = \frac{\partial(m_{\zeta\alpha} x_\beta)}{\partial x_\zeta} + \frac{\partial(m_{\eta\beta} x_\alpha)}{\partial x_\eta} - \frac{1}{2}\frac{\partial^2(m_{\zeta\eta} x_\alpha x_\beta)}{\partial x_\zeta \partial x_\eta} . \qquad \square \ (143)$$

Eq. (141) follows immediately by the Gauss theorem.

There are some immediate consequences of Lemma (3.4):

Corollary 3.2.: 1. If $m_{\alpha\beta} = m^0_{\alpha\beta} = \text{const.} \quad \forall \mathbf{x} \in \partial\Omega$, then

$$\frac{1}{2|\Omega|}\iint_\Omega (m_{\alpha\eta,\eta} x_\beta + m_{\beta\eta,\eta} x_\alpha)\, d\Omega = m^0_{\alpha\beta} - <m_{\alpha\beta}> . \qquad (144)$$

2. If $m_{\alpha\beta} = m^0_{\alpha\beta} = \text{const.}$ and $Q_\alpha = 0 , \quad \forall \mathbf{x} \in \partial\Omega$, then

$$<m_{\alpha\beta}> = m^0_{\alpha\beta} . \qquad (145)$$

Proof:

1. By Eq. (141),

$$\begin{aligned}<m_{\alpha\beta}> &= \frac{1}{2|\Omega|}\oint_{\partial\Omega} \{m_{\zeta\alpha} x_\beta n_\zeta + m_{\eta\beta} x_\alpha n_\eta - m_{\zeta\eta,\eta} x_\alpha x_\beta n_\zeta\}\, dS \\ &= \frac{1}{2|\Omega|}\oint_{\partial\Omega} \{m^0_{\zeta\alpha} x_\beta n_\zeta + m^0_{\eta\beta} x_\alpha n_\eta - m_{\zeta\eta,\eta} x_\alpha x_\beta n_\zeta\}\, dS \\ &= m^0_{\alpha\beta} - \frac{1}{2|\Omega|}\iint_\Omega (m_{\alpha\eta,\eta} x_\beta + m_{\beta\eta,\eta} x_\alpha)\, d\Omega . \end{aligned} \qquad (146)$$

2. Consider $Q_\alpha = m_{\alpha\beta,\beta}$ and $Q_\alpha = 0 , \quad \forall \mathbf{x} \in \partial\Omega$. Equation (145) follows immediately. $\square$

3.2 Elementary variational inequalities and elementary bounds

In what follows, we list some elementary variational inequalities that are based on the principle of minimum potential energy and the principle of minimum complementary energy of the Reissner-Mindlin plate.

The density of the elastic energy of the Reissner-Mindlin plate, $U(\boldsymbol{\chi}, \boldsymbol{\gamma})$, is a quadratic form

$$U(\boldsymbol{\chi}, \boldsymbol{\gamma}) = \frac{1}{2} L_{\alpha\beta\zeta\eta} \chi_{\alpha\beta} \chi_{\zeta\eta} + \frac{1}{2} G_{\alpha\beta} \gamma_\alpha \gamma_\beta \,. \tag{147}$$

For a linear isotropic Reissner-Mindlin plate, it can be explicitly expressed as

$$U(\boldsymbol{\chi}, \boldsymbol{\gamma}) = \frac{D}{2} \left[(1-\nu)\, \chi_{\alpha\beta}\chi_{\alpha\beta} + \nu \chi_{\alpha\alpha}\chi_{\beta\beta}\right] + \frac{G_p}{2} \gamma_\alpha \gamma_\alpha \,. \tag{148}$$

It is evident that the potential energy density function is convex, lower semi-continuous, and proper. The moment tensor, $\mathbf{m} = m_{\alpha\beta}\mathbf{e}_\alpha \otimes \mathbf{e}_\beta$, and the shear force vector, $\mathbf{Q} = Q_\alpha \mathbf{e}_\alpha$, can then be obtained from the constitutive relations

$$m_{\alpha\beta} = \frac{\partial U(\boldsymbol{\chi}, \boldsymbol{\gamma})}{\partial \chi_{\alpha\beta}} \,, \quad \text{or} \quad \mathbf{m} \in \partial_{\boldsymbol{\chi}} U(\boldsymbol{\chi}, \boldsymbol{\gamma}) \,, \tag{149}$$

$$Q_\alpha = \frac{\partial U(\boldsymbol{\chi}, \boldsymbol{\gamma})}{\partial \gamma_\alpha} \,, \quad \text{or} \quad \mathbf{Q} \in \partial_{\boldsymbol{\gamma}} U(\boldsymbol{\chi}, \boldsymbol{\gamma}) \,, \tag{150}$$

where $\partial_{\boldsymbol{\chi}}$ and $\partial_{\boldsymbol{\gamma}}$ are the notations of sub-differential in convex analysis (Ekeland and Temam [7]).

By the Fenchel-Legendre transformation, one may express the density of complementary energy as

$$U^*(\mathbf{m}, \mathbf{Q}) = \sup_{(\boldsymbol{\chi}, \boldsymbol{\gamma})} \{\mathbf{m} : \boldsymbol{\chi} + \mathbf{Q} \cdot \boldsymbol{\gamma} - U(\boldsymbol{\chi}, \boldsymbol{\gamma})\} \,. \tag{151}$$

For linear isotropic thick plates, it is

$$U^*(\mathbf{m}, \mathbf{Q}) = \frac{1}{2D(1-\nu^2)} \left\{(1+\nu)\, m_{\alpha\beta} m_{\alpha\beta} - \nu m_{\alpha\alpha} m_{\beta\beta}\right\} + \frac{1}{2G_p} Q_\alpha Q_\alpha \,, \tag{152}$$

and the constitutive relations inverse to (149) and (150) are

$$\chi_{\alpha\beta} = \frac{\partial U^*(\mathbf{m}, \mathbf{Q})}{\partial m_{\alpha\beta}} \,, \quad \text{or} \quad \boldsymbol{\chi} \in \partial_{\mathbf{m}} U^*(\mathbf{m}, \mathbf{Q}) \,, \tag{153}$$

$$\gamma_\alpha = \frac{\partial U^*(\mathbf{m}, \mathbf{Q})}{\partial Q_\alpha} \,, \quad \text{or} \quad \boldsymbol{\gamma} \in \partial_{\mathbf{Q}} U^*(\mathbf{m}, \mathbf{Q}) \,. \tag{154}$$

Consider the first-type boundary condition $S_u = \partial\Omega$; $S_F = \emptyset$. $\forall\, (\chi_{\alpha\beta}, \gamma_\alpha) \in \mathcal{E}_1(\Omega)$, the plate's potential energy is denoted as

$$\Pi(\boldsymbol{\chi}, \boldsymbol{\gamma}) := \frac{1}{2} \iint_\Omega \left(L_{\alpha\beta\zeta\eta} \chi_{\alpha\beta} \chi_{\zeta\eta} + G_{\alpha\beta} \gamma_\alpha \gamma_\beta\right) d\Omega \,. \tag{155}$$

If $(\chi_{\alpha\beta}, \gamma_\alpha) \in \mathcal{E}_3(\Omega) \subset \mathcal{E}_1(\Omega)$, we denote

$$E(\boldsymbol{\chi}, \boldsymbol{\gamma}) := \frac{1}{2} \iint_\Omega (L_{\alpha\beta\zeta\eta}\chi_{\alpha\beta}\chi_{\zeta\eta} + G_{\alpha\beta}\gamma_\alpha\gamma_\beta)\, d\Omega\,, \quad (\chi_{\alpha\beta}, \gamma_\alpha) \in \mathcal{E}_3(\Omega)\,. \tag{156}$$

The principle of minimum potential energy states that

$$E(\boldsymbol{\chi}, \boldsymbol{\gamma}) = \inf_{(\boldsymbol{\chi},\boldsymbol{\gamma})\in\mathcal{E}_1(\Omega)} \Pi(\boldsymbol{\chi}, \boldsymbol{\gamma})\,. \tag{157}$$

Under the same type of boundary conditions, the complementary energy of the plate is

$$\Gamma(\mathbf{m}, \mathbf{Q}) = \frac{1}{2} \iint_\Omega (N_{\alpha\beta\zeta\eta} m_{\alpha\beta} m_{\zeta\eta} + H_{\alpha\beta} Q_\alpha Q_\beta)\, d\Omega - \oint_{\partial\Omega} (m_{\alpha\beta} n_\beta {\phi_\alpha}^0 + Q_\alpha n_\alpha w^0)\, dS\,, \tag{158}$$

where $(m_{\alpha\beta}, Q_\alpha) \in \mathcal{S}_0(\Omega)$ and $(\phi_\alpha, w) \in \mathcal{D}_1(\Omega)$. By the principle of virtual work, it can be readily shown that

$$\Gamma(\mathbf{m}, \mathbf{Q}) = \frac{1}{2} \iint_\Omega (N_{\alpha\beta\zeta\eta} m_{\alpha\beta} m_{\zeta\eta} + H_{\alpha\beta} Q_\alpha Q_\beta)\, d\Omega - \iint_\Omega (m_{\alpha\beta}\chi^0_{\alpha\beta} + Q_\alpha {\gamma_\alpha}^0)\, d\Omega\,. \tag{159}$$

The principle of minimum complementary energy states that

$$-E(\boldsymbol{\chi}, \boldsymbol{\gamma}) = \sup_{(\mathbf{m},\mathbf{Q})\in\mathcal{S}_0(\Omega)} \Gamma(\mathbf{m}, \mathbf{Q}) \tag{160}$$

or

$$E(\boldsymbol{\chi}, \boldsymbol{\gamma}) = \inf_{(\mathbf{m},\mathbf{Q})\in\mathcal{S}_0(\Omega)} \left\{ \iint_\Omega (m_{\alpha\beta}\chi^0_{\alpha\beta} + Q_\alpha {\gamma_\alpha}^0)\, d\Omega \right.$$
$$\left. - \frac{1}{2} \iint_\Omega (N_{\alpha\beta\zeta\eta} m_{\alpha\beta} m_{\zeta\eta} + H_{\alpha\beta} Q_\alpha Q_\beta)\, d\Omega \right\}, \tag{161}$$

where $(\chi^0_{\alpha\beta}, {\gamma_\alpha}^0) \in \mathcal{E}_1(\Omega)$.

Consider the second-type boundary conditions ($S_u = \emptyset$, $S_F = \partial\Omega$). The overall complementary energy is

$$\Gamma(\mathbf{m}, \mathbf{Q}) = \frac{1}{2} \iint_\Omega (N_{\alpha\beta\zeta\eta} m_{\alpha\beta} m_{\zeta\eta} + H_{\alpha\beta} Q_\alpha Q_\beta)\, d\Omega\,, \quad \forall\, (m_{\alpha\beta}, Q_\alpha) \in \mathcal{S}_1(\Omega)\,. \tag{162}$$

The principle of minimum complementary energy states that

$$E^*(\mathbf{m}, \mathbf{Q}) = \inf_{(\mathbf{m},\mathbf{Q})\in\mathcal{S}_1(\Omega)} \Gamma(\mathbf{m}, \mathbf{Q}) \tag{163}$$

where the minimizer $(\mathbf{m}, \mathbf{Q}) \in \mathcal{S}_1(\Omega)$ ensures that $\chi_{\alpha\beta} = N_{\alpha\beta\zeta\eta} m_{\zeta\eta}$, $\gamma_\alpha = H_{\alpha\beta} Q_\beta$ and $\chi_{\alpha\beta} = \frac{1}{2}(\phi_{\alpha,\beta} + \phi_{\beta,\alpha})$, $\gamma_\alpha = \phi_\alpha + w_{,\alpha}$ and $(\phi_\alpha, w) \in \mathcal{D}_3(\Omega)$.

Accordingly, the potential energy in this case is

$$\begin{aligned}\Pi(\boldsymbol{\chi}, \boldsymbol{\gamma}) &= \frac{1}{2} \iint_\Omega (L_{\alpha\beta\zeta\eta}\chi_{\alpha\beta}\chi_{\zeta\eta} + G_{\alpha\beta}\gamma_\alpha\gamma_\beta)\, d\Omega - \oint_{\partial\Omega} ({Q_n}^0 w + {m_n}^0 \phi_n + m^0_{ns}\phi_s)\, dS \\ &= \frac{1}{2} \iint_\Omega (L_{\alpha\beta\zeta\eta}\chi_{\alpha\beta}\chi_{\zeta\eta} + G_{\alpha\beta}\gamma_\alpha\gamma_\beta)\, d\Omega - \iint_\Omega (m^0_{\alpha\beta}\chi_{\alpha\beta} + {Q_\alpha}^0\gamma_\alpha)\, d\Omega\,,\end{aligned} \tag{164}$$

where $(\chi_{\alpha\beta}, \gamma_\alpha) \in \mathcal{E}_0(\Omega)$ and $(m^0_{\alpha\beta}, {Q_\alpha}^0) \in \mathcal{S}_1(\Omega)$.

The principle of minimum potential energy states that $\Pi(\chi,\gamma)$ attains its minimum, when $\chi_{\alpha\beta}=N_{\alpha\beta\zeta\eta}m^0_{\zeta\eta}$ and $\gamma_\alpha=H_{\alpha\beta}Q_\beta{}^0$ at the value

$$\Pi(\chi,\gamma)=-\Gamma(\mathbf{m},\mathbf{Q})=-E^*(\mathbf{m},\mathbf{Q})\,.$$

In other words

$$E^*(\mathbf{m},\mathbf{Q})=\inf_{(\chi,\gamma)\in\mathcal{E}_0(\Omega)}\left\{\iint_\Omega (m^0_{\alpha\beta}\chi_{\alpha\beta}+Q_\alpha{}^0\gamma_\alpha)\,d\Omega-\frac{1}{2}\iint_\Omega (L_{\alpha\beta\zeta\eta}\chi_{\alpha\beta}\chi_{\zeta\eta}+G_{\alpha\beta}\gamma_\alpha\gamma_\beta)\,d\Omega\right\}. \tag{165}$$

The above elementary variational inequalities have the exact same structures as their counterparts in linear elasticity (see Hill [17]).

Now, we are in a position to discuss the associated elementary bounds. Consider a prescribed rotation/deflection boundary condition such that

$$\phi_\alpha(\mathbf{x})=<\chi_{\alpha\beta}>x_\beta \qquad \forall\mathbf{x}\in\partial\Omega\,, \tag{166}$$

$$w(\mathbf{x})=<\gamma_\alpha>x_\alpha-\frac{1}{2}<\chi_{\alpha\beta}>x_\alpha x_\beta \qquad \forall\mathbf{x}\in\partial\Omega\,. \tag{167}$$

In other words, $(\phi_\alpha,w)\in\mathcal{D}_2(\Omega)\cap\mathcal{D}_3(\Omega)=:D_d(\Omega)$. $\forall\,(\phi_\alpha,w)\in D_d(\Omega)$, we make the following decomposition:

$$\phi_\alpha(\mathbf{x})=<\chi_{\alpha\beta}>x_\beta+\varrho_\alpha(\mathbf{x}) \qquad \forall\mathbf{x}\in\Omega\,, \tag{168}$$

$$w(\mathbf{x})=<\gamma_\alpha>x_\alpha-\frac{1}{2}<\chi_{\alpha\beta}>x_\alpha x_\beta+v(\mathbf{x}) \qquad \forall\mathbf{x}\in\Omega\,. \tag{169}$$

It is obvious that

$$\varrho_\alpha(\mathbf{x})=0 \qquad \forall\mathbf{x}\in\partial\Omega\,, \tag{170}$$

$$\frac{1}{|\Omega|}\iint_\Omega (\varrho_{\alpha,\beta}+\varrho_{\beta,\alpha})\,d\Omega=0\,, \tag{171}$$

$$v(\mathbf{x})=0 \qquad \forall\mathbf{x}\in\partial\Omega\,, \tag{172}$$

$$\frac{1}{|\Omega|}\iint_\Omega (\varrho_\alpha+v_{,\alpha})\,d\Omega=0\,. \tag{173}$$

One may note that from

$$\frac{1}{2|\Omega|}\iint_\Omega (\varrho_{\alpha,\beta}+\varrho_{\beta,\alpha})\,d\Omega=\frac{1}{2|\Omega|}\oint_{\partial\Omega}(\varrho_\alpha n_\beta+\varrho_\beta n_\alpha)\,dS=0 \;\Rightarrow\; \varrho_\alpha=0\,, \quad \forall\mathbf{x}\in\partial\Omega\,. \tag{174}$$

Let $\chi'_{\alpha\beta}=1/2(\varrho_{\alpha,\beta}+\varrho_{\beta,\alpha})$ and $\gamma_\alpha{}'=(\varrho_\alpha+v_{,\alpha})$. One may define a null space of $L^2(\Omega)$ as

$$E_n(\Omega)=\left\{(\chi_{\alpha,\beta},\gamma_\alpha)\in E_0(\Omega)\;\middle|\;\frac{1}{|\Omega|}\iint_\Omega \chi_{\alpha\beta}\,d\Omega=0 \quad\text{and}\quad \frac{1}{|\Omega|}\iint_\Omega \gamma_\alpha\,d\Omega=0\right\} \tag{175}$$

such that $\chi_{\alpha\beta}=\bar{\chi}_{\alpha\beta}+\chi'_{\alpha\beta}$ and $\gamma_\alpha=\bar{\gamma}_\alpha+\gamma_\alpha{}'$ and $(\chi'_{\alpha\beta},\gamma_\alpha{}')\in E_n(\Omega)$. Following Willis [50], by denoting $U(\chi,\gamma)=U(\chi,\chi';\gamma,\gamma')$, the overall potential energy can then be defined as

$$\tilde{\Pi}(\bar{\chi},\bar{\gamma})=\inf_{(\chi',\gamma')\in E_n(\Omega)}\frac{1}{|\Omega|}\iint_\Omega U(\bar{\chi},\chi';\bar{\gamma},\gamma')\,d\Omega\,. \tag{176}$$

For linear elastic plates, one can identify $\tilde{\Pi}(\bar{\chi},\bar{\gamma})$ immediately,

$$\begin{aligned}\frac{1}{|\Omega|}\iint_\Omega U(\bar{\chi},\chi';\bar{\gamma},\gamma')\,d\Omega &= \frac{1}{2|\Omega|}\iint_\Omega \big(L_{\alpha\beta\zeta\eta}(\bar{\chi}_{\alpha\beta}+\chi'_{\alpha\beta})\,(\bar{\chi}_{\zeta\eta}+\chi'_{\zeta\eta})\\ &\quad + G_{\alpha\beta}(\bar{\gamma}_\alpha+\gamma_\alpha{}')\,(\bar{\gamma}_\beta+\gamma_\alpha{}')\big)\,d\Omega\\ &= \frac{1}{2|\Omega|}\iint_\Omega \big(L_{\alpha\beta\zeta\eta}(\bar{\chi}_{\alpha\beta}\bar{\chi}_{\zeta\eta}+\chi'_{\alpha\beta}\chi'_{\zeta\eta})+G_{\alpha\beta}(\bar{\gamma}_\alpha\bar{\gamma}_\beta+\gamma_\alpha{}'\gamma_\beta{}')\big)\,d\Omega\\ &\geq \frac{1}{2|\Omega|}\iint_\Omega (L_{\alpha\beta\zeta\eta}\bar{\chi}_{\alpha\beta}\bar{\chi}_{\zeta\eta}+G_{\alpha\beta}\bar{\gamma}_\alpha\bar{\gamma}_\beta)\,d\Omega\,. \end{aligned} \tag{177}$$

In fact, if $m_{\alpha\beta,\beta}=Q_\alpha$ and $Q_{\alpha,\alpha}=0\ \forall \mathbf{x}\in\Omega$, one can show the following:

Lemma 3.5. For the prescribed displacement boundary condition, if

$$w = <\gamma_\alpha> x_\alpha - \frac{1}{2}<\chi_{\lambda\mu}> x_\lambda x_\mu\,, \qquad \forall \mathbf{x}\in\partial\Omega \tag{178}$$

$$\phi_\alpha = <\chi_{\alpha\beta}> x_\beta\,, \qquad \forall \mathbf{x}\in\partial\Omega\,, \tag{179}$$

then

$$<\mathbf{m}:\boldsymbol{\chi}> + <\mathbf{Q}\cdot\boldsymbol{\gamma}> = <\mathbf{m}>:<\boldsymbol{\chi}> + <\mathbf{Q}>\cdot<\boldsymbol{\gamma}>\,. \tag{180}$$

Proof:

$$\begin{aligned}<\mathbf{m}:\boldsymbol{\chi}> + <\mathbf{Q}\cdot\boldsymbol{\gamma}> &= \frac{1}{|\Omega|}\iint_\Omega (m_{\alpha\beta}\chi_{\alpha\beta}+Q_\alpha\gamma_\alpha)\,d\Omega\\ &= \frac{1}{2|\Omega|}\oint_{\partial\Omega}[m_{\alpha\beta}(\phi_\alpha n_\beta+\phi_\alpha n_\alpha)]\,dS - \frac{1}{|\Omega|}\iint_\Omega (m_{\alpha\beta,\beta}-Q_\alpha)\,\phi_\alpha\,dS\\ &\quad + \frac{1}{|\Omega|}\iint_\Omega Q_\alpha w_{,\alpha}\,d\Omega\\ &= \frac{1}{|\Omega|}\oint_{\partial\Omega} m_{\alpha\beta}(<\chi_{\alpha\gamma}> x_\gamma n_\beta + <\chi_{\beta\gamma}> x_\gamma n_\alpha)\,dS + \frac{1}{|\Omega|}\oint_{\partial\Omega} Q_\alpha w n_\alpha\,dS\\ &= \frac{1}{|\Omega|}\iint_\Omega \big((m_{\alpha\beta,\beta}-Q_\alpha)<\chi_{\alpha\gamma}> x_\gamma + m_{\alpha\beta}<\chi_{\alpha\beta}> + Q_\alpha<\gamma_\alpha>\big)\,d\Omega\\ &= <\mathbf{m}>:<\boldsymbol{\chi}> + <\mathbf{Q}>\cdot<\boldsymbol{\gamma}>\,. \qquad \square \end{aligned} \tag{181}$$

On the other hand, it is still an open problem to find prescribed force boundary conditions under which Eq. (180) holds. The following result holds, with an additional restriction.

Lemma 3.6. For the prescribed resultant boundary conditions, if

$$\iint_\Omega m_{\alpha\beta}x_\gamma\,d\Omega = 0 \tag{182}$$

and $m_{\alpha\beta} = <m_{\alpha\beta}> + <m_{\alpha\beta,\gamma}> x_\gamma$, $Q_\alpha = <Q_\alpha>$, $\forall \mathbf{x}\in\partial\Omega$. Then, Eq. (180) holds.

Proof:

$$\begin{aligned}
\frac{1}{|\Omega|}\iint\limits_{\Omega}(m_{\alpha\beta}\chi_{\alpha\beta}+Q_\alpha\gamma_\alpha)\,d\Omega &= \frac{1}{2|\Omega|}\oint\limits_{\partial\Omega} m_{\alpha\beta}(\phi_\alpha n_\beta+\phi_\beta n_\alpha)\,dS+\frac{1}{|\Omega|}\oint Q_\alpha n_\alpha\, w\, dS\\
&=\frac{1}{2|\Omega|}\oint\limits_{\partial\Omega}\left[<m_{\alpha\beta}>+<m_{\alpha\beta,\gamma}>x_\gamma\right](\phi_\alpha n_\beta+\phi_\beta n_\alpha)\,dS\\
&\quad+\frac{1}{|\Omega|}\oint\limits_{\partial\Omega}<Q_\alpha>n_\alpha\, w\, dS\\
&=<m_{\alpha\beta}><\chi_{\alpha\beta}>+<Q_\alpha><\gamma_\alpha>\\
&\quad+\frac{1}{|\Omega|}\iint\limits_{\Omega}<m_{\alpha\beta,\gamma}>x_\gamma\chi_{\alpha\beta}\,d\Omega\\
&=<m_{\alpha\beta}><\chi_{\alpha\beta}>+<Q_\alpha><\gamma_\alpha>\ . \qquad (183)
\end{aligned}$$

In the last step, we assume that $\exists\chi^0_{\zeta\eta,\gamma}$, such that

$$<m_{\alpha\beta,\gamma}>=N_{\alpha\beta\zeta\eta}\chi^0_{\zeta\eta,\gamma}\,, \qquad (184)$$

then

$$\frac{1}{|\Omega|}\iint\limits_{\Omega}<m_{\alpha\beta,\gamma}>x_\gamma\chi_{\alpha\beta}\,d\Omega=\frac{1}{|\Omega|}\iint\limits_{\Omega}\chi^0_{\zeta\eta,\gamma}x_\gamma m_{\alpha\beta}\,d\Omega=0\,.$$ □

In general, for Reissner-Mindlin plates, we have the following results:

Proposition 3.1. Suppose $m_{\alpha\beta,\beta}-Q_\alpha=0$ and $Q_{\alpha,\alpha}=0$. The following identities hold:

1.
$$\begin{aligned}
&\frac{1}{|\Omega|}\oint\limits_{\partial\Omega}(m_{\alpha\beta}n_\alpha-<m_{\alpha\beta}>n_\alpha)(\phi_\alpha-<\phi_\alpha>)\,dS\\
&+\frac{1}{|\Omega|}\oint\limits_{\partial\Omega}(Q_\alpha n_\alpha-<Q_\alpha>n_\alpha)(w-<w_{,\zeta}>x_\zeta)\,dS\\
&=<\mathbf{m}:\boldsymbol{\chi}>+<\mathbf{Q}\cdot\boldsymbol{\gamma}>-<\mathbf{m}>:<\boldsymbol{\chi}>-<\mathbf{Q}>\cdot<\boldsymbol{\gamma}>\,. \qquad (185)
\end{aligned}$$

2.
$$\begin{aligned}
&\frac{1}{|\Omega|}\oint\limits_{\partial\Omega}(m_{\alpha\beta}n_\beta-<m_{\alpha\beta}>n_\beta)(\phi_\alpha-<\chi_{\alpha\gamma}>x_\gamma)\,dS\\
&+\frac{1}{|\Omega|}\oint\limits_{\partial\Omega}Q_\alpha n_\alpha(w-<\gamma_\zeta>x_\zeta+\frac{1}{2}<\chi_{\lambda\mu}>x_\lambda x_\mu)\,dS\\
&=<\mathbf{m}:\boldsymbol{\chi}>+<\mathbf{Q}\cdot\boldsymbol{\gamma}>-<\mathbf{m}>:<\boldsymbol{\chi}>-<\mathbf{Q}>\cdot<\boldsymbol{\gamma}>\,. \qquad (186)
\end{aligned}$$

The proof of the proposition is a direct use of the Gauss theorem and the integration by parts. □

Then, the overall potential energy under the prescribed deflection boundary condition is

$$\tilde{\Pi}(\bar{\boldsymbol{\chi}},\bar{\boldsymbol{\gamma}})=\frac{1}{2}(<\mathbf{m}>:<\boldsymbol{\chi}>+<\mathbf{Q}>\cdot<\boldsymbol{\gamma}>)\,. \qquad (187)$$

Obviously,

$$\bar{m}_{\alpha\beta} = \frac{\partial \tilde{\Pi}(\bar{\boldsymbol{\chi}}, \bar{\boldsymbol{\gamma}})}{\partial \bar{\chi}_{\alpha\beta}}, \quad \text{or} \quad \bar{\mathbf{m}} \in \partial_{\bar{\boldsymbol{\chi}}} \tilde{\Pi}(\bar{\boldsymbol{\chi}}, \bar{\boldsymbol{\gamma}}), \tag{188}$$

$$\bar{Q}_{\alpha} = \frac{\partial \tilde{\Pi}(\bar{\boldsymbol{\chi}}, \bar{\boldsymbol{\gamma}})}{\partial \bar{\gamma}_{\alpha}}, \quad \text{or} \quad \bar{\mathbf{Q}} \in \partial_{\bar{\boldsymbol{\gamma}}} \tilde{\Pi}(\bar{\boldsymbol{\chi}}, \bar{\boldsymbol{\gamma}}). \tag{189}$$

The inequality (157) can then be modified as

$$\tilde{\Pi}(\bar{\boldsymbol{\chi}}, \bar{\boldsymbol{\gamma}}) = \inf_{(\boldsymbol{\chi}', \boldsymbol{\gamma}') \in E_n(\Omega)} \Pi(\bar{\boldsymbol{\chi}}, \boldsymbol{\chi}'; \bar{\boldsymbol{\gamma}}, \boldsymbol{\gamma}') \le \overline{\Pi(\bar{\boldsymbol{\chi}}, \bar{\boldsymbol{\gamma}})} = \sum_{i=1}^{n} c_i \Pi_i(\bar{\boldsymbol{\chi}}, \bar{\boldsymbol{\gamma}}), \tag{190}$$

where $c_i = h|\Omega_i|/h|\Omega| = |\Omega_i|/|\Omega|$, and $\Pi_i(\bar{\boldsymbol{\chi}}, \bar{\boldsymbol{\gamma}}) := \iint_{\Omega_i} U(\bar{\boldsymbol{\chi}}, \bar{\boldsymbol{\gamma}})\, d\Omega$. Equation (190) is the well-known Voigt bound.

For the second-type boundary problem, the prescribed force boundary conditions $Q_n = \hat{Q}_n$, $M_{ns} = \hat{M}_{ns}$ and $M_n = \hat{M}_n$ are chosen such that they are compatible with the special boundary value of the moments and resultants,

$$m_{\alpha\beta}(\mathbf{x}) = <m_{\alpha\beta}> + \sigma^0_{\alpha\beta}(\mathbf{x}) \qquad \forall \mathbf{x} \in \partial\Omega, \tag{191}$$

$$Q_{\alpha}(\mathbf{x}) = <Q_{\alpha}> \qquad \forall \mathbf{x} \in \partial\Omega, \tag{192}$$

with

$$\sigma^0_{\zeta\alpha} x_{\beta} + \sigma^0_{\zeta\beta} x_{\alpha} = <Q_{\zeta}>\, x_{\alpha} x_{\beta}. \tag{193}$$

For $(m_{\alpha\beta}, Q_{\alpha}) \in \mathcal{S}_1(\Omega) \cap \mathcal{S}_2(\Omega)$, we make the decomposition

$$m_{\alpha\beta} = <m_{\alpha\beta}> + \sigma_{\alpha\beta}, \tag{194}$$

$$Q_{\alpha} = <Q_{\alpha}> + \tau_{\alpha}. \tag{195}$$

By virtue of Eqs. (141) and (140), it can be readily shown that

$$\sigma^0_{\zeta\alpha} x_{\beta} + \sigma^0_{\zeta\beta} x_{\alpha} = <Q_{\alpha}>\, x_{\alpha} x_{\beta} \qquad \forall \mathbf{x} \in \partial\Omega \quad \Rightarrow \quad \frac{1}{|\Omega|} \iint_{\Omega} \sigma_{\alpha\beta}\, d\Omega = 0, \tag{196}$$

$$\tau_{\alpha} = 0 \qquad \forall \mathbf{x} \in \partial\Omega \quad \Rightarrow \quad \frac{1}{|\Omega|} \iint_{\Omega} \tau_{\alpha}\, d\Omega = 0. \tag{197}$$

Thus, $(\sigma_{\alpha\beta}, \tau_{\alpha}) \in S_n$, where

$$S_n(\Omega) := \Bigg\{ (\sigma_{\beta}, \tau_{\alpha}) \,\Big|\, \sigma_{\alpha\beta,\beta} - \tau_{\alpha} = <Q_{\alpha}>, \quad \tau_{\alpha,\alpha} = 0, \quad \frac{1}{|\Omega|} \iint_{\Omega} \sigma_{\alpha\beta}\, d\Omega = 0, \quad \text{and} \quad \frac{1}{|\Omega|} \iint_{\Omega} \tau_{\alpha}\, d\Omega = 0 \Bigg\}. \tag{198}$$

Accordingly, one can define the overall complementary energy potential as

$$\tilde{\Gamma}(\bar{\mathbf{m}}, \bar{\mathbf{Q}}) := \inf_{(\boldsymbol{\sigma}, \boldsymbol{\tau}) \in S_n(\Omega)} \Gamma(\bar{\mathbf{m}}, \boldsymbol{\sigma}; \bar{\mathbf{Q}}, \boldsymbol{\tau}). \tag{199}$$

Again, for a linear elastic plate, we can identify $\tilde{\Gamma}(\bar{\mathbf{m}},\bar{\mathbf{Q}})$ as

$$\begin{aligned}\Gamma(\bar{\mathbf{m}},\boldsymbol{\sigma};\bar{\mathbf{Q}},\boldsymbol{\tau}) &= \frac{1}{2}\iint\limits_{\Omega}\big(N_{\alpha\beta\zeta\eta}(\bar{m}_{\alpha\beta}+\sigma_{\alpha\beta})\,(\bar{m}_{\zeta\eta}+\sigma_{\zeta\eta})+H_{\alpha\beta}(\bar{Q}_{\alpha}+\tau_{\alpha})\,(\bar{Q}_{\beta}+\tau_{\beta})\big)\,d\Omega\\ &= \frac{1}{2}\iint\limits_{\Omega}(N_{\alpha\beta\zeta\eta}\bar{m}_{\alpha\beta}\bar{m}_{\zeta\eta}+N_{\alpha\beta\zeta\eta}\sigma_{\alpha\beta}\sigma_{\zeta\eta}+H_{\alpha\beta}\bar{Q}_{\alpha}\bar{Q}_{\beta}+H_{\alpha\beta}\tau_{\alpha}\tau_{\beta})\,d\Omega\\ &\geq \frac{1}{2}\iint\limits_{\Omega}(N_{\alpha\beta\zeta\eta}\bar{m}_{\alpha\beta}\bar{m}_{\zeta\eta}+H_{\alpha\beta}\bar{Q}_{\alpha}\bar{Q}_{\beta})\,d\Omega\,.\end{aligned} \tag{200}$$

Hence,

$$\bar{\chi}_{\alpha\beta}=\frac{\partial\tilde{\Gamma}(\bar{\mathbf{m}},\bar{\mathbf{Q}})}{\partial\bar{m}_{\alpha\beta}}\,,\quad\text{or}\quad\bar{\boldsymbol{\chi}}\in\partial_{\bar{\mathbf{m}}}\tilde{\Gamma}(\bar{\mathbf{m}},\bar{\mathbf{Q}})\,, \tag{201}$$

$$\bar{\gamma}_{\alpha}=\frac{\partial\tilde{\Gamma}(\bar{\mathbf{m}},\bar{\mathbf{Q}})}{\partial\bar{Q}_{\alpha}}\,,\quad\text{or}\quad\bar{\boldsymbol{\gamma}}\in\partial_{\bar{\mathbf{Q}}}\tilde{\Gamma}(\bar{\mathbf{m}},\bar{\mathbf{Q}})\,, \tag{202}$$

and by the transformation

$$\tilde{\Gamma}^{*}(\bar{\boldsymbol{\chi}},\bar{\boldsymbol{\gamma}})=\sup_{(\bar{\mathbf{m}},\bar{\mathbf{Q}})\in\mathrm{Sym}_0}\{\bar{\mathbf{m}}:\bar{\boldsymbol{\chi}}+\bar{\mathbf{Q}}\cdot\bar{\boldsymbol{\gamma}}-\tilde{\Gamma}(\bar{\mathbf{m}},\bar{\mathbf{Q}})\} \tag{203}$$

one will have

$$\bar{m}_{\alpha\beta}=\frac{\partial\tilde{\Gamma}^{*}(\bar{\boldsymbol{\chi}},\bar{\boldsymbol{\gamma}})}{\partial\bar{\chi}_{\alpha\beta}}\,,\quad\text{or}\quad\bar{\mathbf{m}}\in\partial_{\bar{\boldsymbol{\chi}}}\tilde{\Gamma}^{*}(\bar{\boldsymbol{\chi}},\bar{\boldsymbol{\gamma}})\,, \tag{204}$$

$$\bar{Q}_{\alpha}=\frac{\partial\tilde{\Gamma}^{*}(\bar{\boldsymbol{\chi}},\bar{\boldsymbol{\gamma}})}{\partial\bar{\gamma}_{\alpha}}\,,\quad\text{or}\quad\bar{\mathbf{Q}}\in\partial_{\bar{\boldsymbol{\gamma}}}\tilde{\Gamma}^{*}(\bar{\boldsymbol{\chi}},\bar{\boldsymbol{\gamma}})\,, \tag{205}$$

The variational inequality (199) furnishes the estimate

$$\tilde{\Gamma}(\bar{\mathbf{m}},\bar{\mathbf{Q}})=\inf_{(\boldsymbol{\sigma},\boldsymbol{\tau})\in\mathcal{S}_n(\Omega)}\Gamma(\bar{\mathbf{m}},\boldsymbol{\sigma};\bar{\mathbf{Q}},\boldsymbol{\tau})\leq\overline{\Gamma(\bar{\mathbf{m}},\bar{\mathbf{Q}})}=\sum_{i=1}^{n}c_i\Gamma_i(\bar{\mathbf{m}},\bar{\mathbf{Q}})\,, \tag{206}$$

where $\Gamma_i(\bar{\mathbf{m}},\bar{\mathbf{Q}})=\iint\limits_{\Omega_i}U^{*}(\bar{\mathbf{m}},\bar{\mathbf{Q}})\,d\Omega$. Equation (206) is the well-known Reuss bound. For the second-type boundary condition, let

$$\tilde{\Pi}(\bar{\boldsymbol{\chi}},\bar{\boldsymbol{\gamma}})=\sup_{(\mathbf{m},\mathbf{Q})\in\mathcal{S}_1(\Omega)}\left\{\frac{1}{|\Omega|}\iint\limits_{\Omega}(\mathbf{m}:\bar{\boldsymbol{\chi}}+\mathbf{Q}\cdot\bar{\boldsymbol{\gamma}})\,d\Omega-\frac{1}{|\Omega|}\iint\limits_{\Omega}U^{*}(\mathbf{m},\mathbf{Q})\,d\Omega\right\}. \tag{207}$$

The inequality (206) implies

$$\begin{aligned}\tilde{\Pi}(\bar{\boldsymbol{\chi}},\bar{\boldsymbol{\gamma}})\geq\tilde{\Pi}^{**}(\bar{\boldsymbol{\chi}},\bar{\boldsymbol{\gamma}})\geq\sup_{(\mathbf{m}'\mathbf{Q}')\in E_n{}^{*}(\Omega)}\ \inf_{(\boldsymbol{\chi}'\boldsymbol{\gamma}')\in E_n(\Omega)}&\{(\bar{\mathbf{m}},\bar{\boldsymbol{\chi}})+(\mathbf{m}',\boldsymbol{\chi}')\\ &+(\bar{\mathbf{Q}},\bar{\boldsymbol{\gamma}})+(\mathbf{Q}',\boldsymbol{\gamma}')-\Pi^{*}(\bar{\mathbf{m}},\mathbf{m}';\bar{\mathbf{Q}},\mathbf{Q}')\}\end{aligned} \tag{208}$$

which is similar to the variational inequality shown by Willis [50] in a general duality framework. The inequality of the minimum potential energy (165) renders the corresponding upper bound

$$\tilde{\Gamma}(\bar{\mathbf{m}}, \bar{\mathbf{Q}}) = \sup_{(\chi', \gamma') \in \bar{S}_n^*(\Omega)} \left\{ \iint_{\Omega} \left(\bar{\mathbf{m}} : (\bar{\chi} + \chi') + \bar{\mathbf{Q}} \cdot (\bar{\gamma} + \gamma') \right) d\Omega \right.$$
$$\left. -\frac{1}{2} \iint_{\Omega} \left((\bar{\chi} + \chi') : \mathbf{L} : (\bar{\chi} + \chi') + (\bar{\gamma} + \gamma') \cdot \mathbf{G} \cdot (\bar{\gamma} + \gamma') \right) d\Omega \right\}. \tag{209}$$

3.3 Hashin-Shtrikman/Talbot-Willis type principles

In the following, two comparison variational principles of Hashin-Shtrikman type (Hashin and Shtrikman [13], Hill [17]) are presented for the Reissner-Mindlin plate. For the sake of updated documentation, the style and notations of the presentation follow largely from Talbot and Willis [38], which, in the author's opinion, represent the contemporary standard in the treatment of the comparison variational principles. In the contex of a nonlinear continuum, they are often referred to as Talbot-Willis variational principles in the literature.

Consider the first type (prescribed deflection/rotation) boundary-value problem. Let $(\phi_\alpha, w) \in \mathcal{D}_1(\Omega)$ be a special kinematically admissible deflection field, which is the superposition

$$\phi_\alpha(\mathbf{x}) = \phi_\alpha{}^0(\mathbf{x}) + \phi_\alpha{}^1(\mathbf{x}), \tag{210}$$

$$w(\mathbf{x}) = w^0(\mathbf{x}) + w^1(\mathbf{x}), \tag{211}$$

such that $(\phi_\alpha{}^0, w^0)$ is a solution under the first-type boundary-value problem in the comparison plate, which has the elastic stiffness $L^0_{\alpha\beta\zeta\eta}, G^0_{\alpha\beta}$; and $\left(\phi_\alpha{}^1(\mathbf{x}), w^1(\mathbf{x})\right) \in C_0{}^1(\Omega)$. Accordingly,

$$\chi_{\alpha\beta} = \chi^0_{\alpha\beta} + \chi^1_{\alpha\beta}, \tag{212}$$

$$\gamma_\alpha = \gamma_\alpha{}^0 + \gamma_\alpha{}^1, \tag{213}$$

where $\{\chi^1_{\alpha\beta}, \gamma_\alpha{}^1\} := \{1/2(\phi^1_{\alpha,\beta} + \phi^1_{\beta,\alpha}), (\phi_\alpha{}^1 + w^1_{,\alpha})\} \in B$, a closed subspace of $[L^2(\Omega)]^3 \times [L^2(\Omega)]^2$. In addition, that $(\phi_\alpha{}^0, w^0)$ is kinematically admissible, for the comparison plate, we have

$$m^0_{\alpha\beta} = L^0_{\alpha\beta\zeta\eta}\chi^0_{\zeta\eta}, \qquad Q_\alpha{}^0 = G^0_{\alpha\beta}\gamma_\beta{}^0, \tag{214}$$

$$m^0_{\alpha\beta,\beta} - Q_\alpha{}^0 = 0, \qquad Q^0_{\alpha,\alpha} = 0. \tag{215}$$

We are looking for the solution of the following optimization problem:

$$\text{(The primal problem)} \quad \mathcal{P}: \inf_{(\chi^1, \gamma^1) \in B} \Pi(\chi^1, \gamma^1), \tag{216}$$

where

$$\Pi(\chi^1, \gamma^1) := \iint_{\Omega} U(\chi^0 + \chi^1; \gamma^0 + \gamma^1)\, d\Omega$$
$$= \frac{1}{2} \iint_{\Omega} \{L_{\alpha\beta\zeta\eta}(\chi^0_{\alpha\beta} + \chi^1_{\alpha\beta})(\chi^0_{\zeta\eta} + \chi^1_{\zeta\eta}) + G_{\alpha\beta}(\gamma_\alpha{}^0 + \gamma_\alpha{}^1)(\gamma_\beta{}^0 + \gamma_\beta{}^1)\}\, d\Omega. \tag{217}$$

For $(\mathbf{m}^*, \mathbf{Q}^*) \in B^*$, the dual space of B, define the dual potential $\Pi^*(\mathbf{m}^*, \mathbf{Q}^*)$,

$$\Pi^*(\mathbf{m}^*, \mathbf{Q}^*) = \sup_{(\boldsymbol{\chi}^1, \boldsymbol{\gamma}^1) \in B} \{(\mathbf{m}^*, \boldsymbol{\chi}^1) + (\mathbf{Q}^*, \boldsymbol{\gamma}^1) - \Pi(\boldsymbol{\chi}^1, \boldsymbol{\gamma}^1)\}. \tag{218}$$

We say $(\mathbf{m}^*, \mathbf{Q}^*) \in B^0$, the set of annihilators of B, if

$$(\mathbf{m}^*, \boldsymbol{\chi}^1) + (\mathbf{Q}^*, \boldsymbol{\gamma}^1) = \iint_\Omega (m^*_{\alpha\beta}\chi^1_{\alpha\beta} + Q_\alpha{}^*\gamma_\alpha{}^1)\, d\Omega = 0, \tag{219}$$

which posts additional constraints on $(\mathbf{m}^*, \mathbf{Q}^*)$ (see Lemma (2.1)).

Subsequently,

$$\Pi(\boldsymbol{\chi}^1, \boldsymbol{\gamma}^1) + \Pi^*(\mathbf{m}^*, \mathbf{Q}^*) \geq 0 \qquad \forall\, \mathbf{m}^*, \mathbf{Q}^* \in B^0; \tag{220}$$

it then implies

$$-\Pi^*(\mathbf{m}^*, \mathbf{Q}^*) \leq \inf_{(\boldsymbol{\chi}^1; \boldsymbol{\gamma}^1) \in B} \mathcal{P} \leq \Pi(\boldsymbol{\chi}^1, \boldsymbol{\gamma}^1). \tag{221}$$

This suggests a dual problem:

$$\text{(The dual problem)} \quad \mathcal{P}^* : \sup_{(\mathbf{m}^*, \mathbf{Q}^*) \in B^0} \{-\Pi^*(\mathbf{m}^*, \mathbf{Q}^*)\}. \tag{222}$$

For practical purposes, instead of optimizing the dual potential in a prescribed displacement boundary-value problem, it is convenient to consider a different boundary value problem: optimize the complementary potential energy under the prescribed force boundary-value problem. Define

$$\begin{aligned} \Gamma(\mathbf{m}^1, \mathbf{Q}^1) &= \iint_\Omega U^*(\mathbf{m}^0 + \mathbf{m}^1; \mathbf{Q}^0 + \mathbf{Q}^1)\, d\Omega \\ &= \frac{1}{2} \iint_\Omega \{N_{\alpha\beta\zeta\eta}(m^0_{\alpha\beta} + m^1_{\alpha\beta})\,(m^0_{\zeta\eta} + m^1_{\zeta\eta}) \\ &\quad + H_{\alpha\beta}(Q_\alpha{}^0 + Q_\alpha{}^1)\,(Q_\beta{}^0 + Q_\beta{}^1)\}\, d\Omega, \end{aligned} \tag{223}$$

where $\mathbf{m} = \mathbf{m}^0 + \mathbf{m}^1$ and $\mathbf{Q} = \mathbf{Q}^0 + \mathbf{Q}^1$ are a special statically admissible resultant field in which $(\mathbf{m}^0, \mathbf{Q}^0)$ is the solution of the second-type boundary problem in the comparison plate, and $(\mathbf{m}^1, \mathbf{Q}^1)$ is the perturbation, such that $(\mathbf{m}^1, \mathbf{Q}^1) \in H(\Omega) \subset [L_0{}^2(\Omega)]^3 \times [L_0{}^2(\Omega)]^2$.

Consider the optimization problem

$$\mathcal{P}_d : \inf_{(\mathbf{m}^1, \mathbf{Q}^1) \in H(\Omega)} \Gamma(\mathbf{m}^1, \mathbf{Q}^1). \tag{224}$$

It has a duality approach too, i.e. $\forall\, (\boldsymbol{\chi}^*, \boldsymbol{\gamma}^*) \in H^0$, the annihilator set of H, i.e.,

$$H^0 := \left\{(\boldsymbol{\chi}^*, \boldsymbol{\gamma}^*) \mid \iint_\Omega (m^1_{\alpha\beta}\chi^*_{\alpha\beta} + Q_\alpha{}^1\gamma_\alpha{}^*)\, d\Omega = 0, \qquad \forall\, (\mathbf{m}^1, \mathbf{Q}^1) \in H\right\}. \tag{225}$$

The restriction on $(\chi^*_{\alpha\beta}, \gamma_\alpha{}^*)$ is stated in the Lemma (2.2).

Subsequently, we have

$$-\Gamma^*(\boldsymbol{\chi}^*, \boldsymbol{\gamma}^*) \leq \inf_{(\mathbf{m}^1, \mathbf{Q}^1) \in H(\Omega)} \mathcal{P}_d \leq \Gamma(\mathbf{m}^1, \mathbf{Q}^1), \tag{226}$$

where

$$\Gamma^*(\boldsymbol{\chi}^*,\boldsymbol{\gamma}^*) := \sup_{(\mathbf{m}^1,\mathbf{Q}^1)\in H} \{(\mathbf{m}^1,\boldsymbol{\chi}^*) + (\mathbf{Q}^1,\boldsymbol{\gamma}^*) - \Gamma(\mathbf{m}^1,\mathbf{Q}^1)\}. \tag{227}$$

Remark 3.1. The conditions that define the annihilator sets of B and H, Eqs. (219) and (225), have important physical interpretations. Equation (219) is related to the principle of virtual work,

$$\iint_\Omega (m^*_{\alpha\beta}\chi^1_{\alpha\beta} + Q_\alpha{}^*\gamma_\alpha{}^1)\, d\Omega = \iint_\Omega [m^*_{\alpha\beta}(\chi_{\alpha\beta} - \chi^0_{\alpha\beta}) + Q_\alpha{}^*(\gamma_\alpha - \gamma_\alpha{}^0)]\, d\Omega = 0\,, \tag{228}$$

and Eq. (225) is related to the virtual complementary work

$$\iint_\Omega (\chi^*_{\alpha\beta}m^1_{\alpha\beta} + \gamma_\alpha{}^*Q_\alpha{}^1)\, d\Omega = \iint_\Omega [\chi^*_{\alpha\beta}(m_{\alpha\beta} - m^0_{\alpha\beta}) + \gamma_\alpha{}^*(Q_\alpha - Q_\alpha{}^0)]\, d\Omega = 0\,, \qquad \square \tag{229}$$

Now, we are in the position to describe the Hashin-Shtrikman/Talbot-Willis principle. Introduce two comparison functionals

$$\begin{aligned}\Pi_0(\boldsymbol{\chi}^1,\boldsymbol{\gamma}^1) \quad (\text{or} \quad \Pi^0(\boldsymbol{\chi}^1,\boldsymbol{\gamma}^1)) : &= \frac{1}{2}\iint_\Omega U_0(\boldsymbol{\chi}^0 + \boldsymbol{\chi}^1; \boldsymbol{\gamma}^0 + \boldsymbol{\gamma}^1)\, d\Omega \\ &= \frac{1}{2}\iint_\Omega \{L^0_{\alpha\beta\zeta\eta}(\chi^0_{\alpha\beta} + \chi^1_{\alpha\beta})\,(\chi^0_{\zeta\eta} + \chi^1_{\zeta\eta}) \\ &\quad + G^0_{\alpha\beta}(\gamma_\alpha{}^0 + \gamma_\alpha{}^1)\,(\gamma_\beta{}^0 + \gamma_\beta{}^1)\}\, d\Omega\end{aligned} \tag{230}$$

if $L_{\alpha\beta\zeta\eta} - L^0_{\alpha\beta\zeta\eta} > 0$, and $G_{\alpha\beta} - G^0_{\alpha\beta} > 0$, or $L_{\alpha\beta\zeta\eta} - L^0_{\alpha\beta\zeta\eta} < 0$, and $G_{\alpha\beta} - G^0_{\alpha\beta} < 0$.

The associated potential differences are defined as

$$\begin{aligned}&\underline{F}(\boldsymbol{\chi}^1,\boldsymbol{\gamma}^1) := \Pi(\boldsymbol{\chi}^1,\boldsymbol{\gamma}^1) - \Pi_0(\boldsymbol{\chi}^1,\boldsymbol{\gamma}^1)\,, \\ &\Delta L_{\alpha\beta\zeta\eta} := L_{\alpha\beta\zeta\eta} - L^0_{\alpha\beta\zeta\eta} > 0\,, \quad \Delta G_{\alpha\beta} := G_{\alpha\beta} - G^0_{\alpha\beta} > 0\,,\end{aligned} \tag{231}$$

$$\begin{aligned}&\overline{F}(\boldsymbol{\chi}^1,\boldsymbol{\gamma}^1) := \Pi(\boldsymbol{\chi}^1,\boldsymbol{\gamma}^1) - \Pi^0(\boldsymbol{\chi}^1,\boldsymbol{\gamma}^1)\,, \\ &\Delta L_{\alpha\beta\zeta\eta} := L_{\alpha\beta\zeta\eta} - L^0_{\alpha\beta\zeta\eta} < 0\,, \quad \Delta G_{\alpha\beta} := G_{\alpha\beta} - G^0_{\alpha\beta} < 0\,.\end{aligned} \tag{232}$$

Introduce the moment polarization tensor, $\sigma_{\alpha\beta}$, and the shear force polarization vector, τ_α, such that

$$m_{\alpha\beta} = L^0_{\alpha\beta\zeta\eta}\chi_{\zeta\eta} + \sigma_{\alpha\beta}\,, \quad \text{or} \quad \sigma_{\alpha\beta} = (L_{\alpha\beta\zeta\eta} - L^0_{\alpha\beta\zeta\eta})\,\chi_{\zeta\eta}\,, \tag{233}$$

$$Q_\alpha = G^0_{\alpha\beta}\gamma_\beta + \tau_\alpha\,, \quad \text{or} \quad \tau_\alpha = (G_{\alpha\beta} - G^0_{\alpha\beta})\,\gamma_\beta\,. \tag{234}$$

By the transformation,

$$\underline{F}^*(\boldsymbol{\sigma},\boldsymbol{\tau}) = \sup_{(\boldsymbol{\chi}',\boldsymbol{\gamma}')\in B} \{(\boldsymbol{\sigma},\boldsymbol{\chi}^1) + (\boldsymbol{\tau},\boldsymbol{\gamma}^1) - \underline{F}(\boldsymbol{\chi}^1,\boldsymbol{\gamma}^1)\}\,, \tag{235}$$

$$\overline{F^*}(\boldsymbol{\sigma},\boldsymbol{\tau}) = \inf_{(\boldsymbol{\chi}',\boldsymbol{\gamma}')\in B} \{(\boldsymbol{\sigma},\boldsymbol{\chi}^1) + (\boldsymbol{\tau},\boldsymbol{\gamma}^1) - \overline{F}(\boldsymbol{\chi}^1,\boldsymbol{\gamma}^1)\}\,. \tag{236}$$

Note that the supremum in Eq. (235) and the infimum in Eq. (236) are attained when Eqs. (233) and (234) hold. It can be readily shown that the primal problem can be realized by the following comparison strategy:

$$\begin{aligned}
&\inf_{(\chi^1,\gamma^1)\in B}\{(\boldsymbol{\sigma},\boldsymbol{\chi}^1)+(\boldsymbol{\tau},\boldsymbol{\gamma}^1)+\Pi_0(\boldsymbol{\chi}^1,\boldsymbol{\gamma}^1)\}-\underline{F^*}(\boldsymbol{\sigma},\boldsymbol{\tau})\,,\\
&\qquad\qquad \le \inf\mathcal{P}\le\\
&\inf_{(\chi^1,\gamma^1)\in B}\{(\boldsymbol{\sigma},\boldsymbol{\chi}^1)+(\boldsymbol{\tau},\boldsymbol{\gamma}^1)+\Pi^0(\boldsymbol{\chi}^1,\boldsymbol{\gamma}^1)\}-\overline{F^*}(\boldsymbol{\sigma},\boldsymbol{\tau})\,.
\end{aligned}\tag{237}$$

Substituting $\chi^1_{\alpha\beta}=\Delta L^{-1}_{\alpha\beta\zeta\eta}\sigma_{\zeta\eta}-\chi^0_{\alpha\beta}$ and $\gamma_\alpha{}^1=\Delta G^{-1}_{\alpha\beta}\tau_\beta-\gamma_\alpha{}^0$ into Eqs. (235) and (236), we have

$$\begin{aligned}
\underline{F^*}(\boldsymbol{\sigma},\boldsymbol{\tau})\ \left(\text{or }\overline{F^*}(\boldsymbol{\sigma},\boldsymbol{\tau})\right)\ =&\iint_\Omega\Big\{\sigma_{\alpha\beta}\chi^1_{\alpha\beta}-\frac{1}{2}\,\Delta L_{\alpha\beta\zeta\eta}\chi^1_{\alpha\beta}\chi^1_{\alpha\beta}\\
&-\frac{1}{2}\,\Delta L_{\alpha\beta\zeta\eta}\chi^0_{\alpha\beta}\chi^0_{\zeta\eta}-L_{\alpha\beta\zeta\eta}\chi^0_{\alpha\beta}\chi^1_{\zeta\eta}+L^0_{\alpha\beta\zeta\eta}\chi_{\alpha\beta}\chi^1_{\zeta\eta}\\
&+\tau_\alpha\gamma_\alpha{}^1-\frac{1}{2}\,\Delta G_{\alpha\beta}\gamma_\alpha{}^1\gamma_\beta{}^1-\frac{1}{2}\,\Delta G_{\alpha\beta}\gamma_\alpha{}^0\gamma_\beta{}^0\\
&-G_{\alpha\beta}\gamma_\alpha{}^0\gamma_\beta{}^1+G^0_{\alpha\beta}\gamma_\alpha\gamma_\beta{}^1\Big\}\,d\Omega\\
=&\frac{1}{2}\iint_\Omega(\Delta L^{-1}_{\alpha\beta\zeta\eta}\sigma_{\alpha\beta}\sigma_{\zeta\eta}+\Delta G^{-1}_{\alpha\beta}\tau_\alpha\tau_\beta-2\sigma_{\alpha\beta}\chi^0_{\alpha\beta}-2\tau_\alpha\gamma_\alpha{}^0)\,d\Omega\,.
\end{aligned}\tag{238}$$

Compute

$$\underline{I}:=\inf_{(\chi^1,\gamma^1)\in B}\{(\boldsymbol{\sigma},\boldsymbol{\chi}^1)+(\boldsymbol{\tau},\boldsymbol{\gamma}^1)+\Pi_0(\boldsymbol{\chi}^1,\boldsymbol{\gamma}^1)\}-\underline{F^*}(\boldsymbol{\sigma},\boldsymbol{\tau})\,,\tag{239}$$

$$\overline{I}:=\inf_{(\chi^1,\gamma^1)\in B}\{(\boldsymbol{\sigma},\boldsymbol{\chi}^1)+(\boldsymbol{\tau},\boldsymbol{\gamma}^1)+\Pi^0(\boldsymbol{\chi}^1,\boldsymbol{\gamma}^1)\}-\overline{F^*}(\boldsymbol{\sigma},\boldsymbol{\tau})\,.\tag{240}$$

The infima are attained when (recall and compare with Remark 3.1)

$$\iint_\Omega\{(L^0_{\alpha\beta\zeta\eta}\chi_\zeta\eta^1+\sigma_{\alpha\beta})\,\chi^1_{\alpha\beta}+(G^0_{\alpha\beta}\gamma_\beta{}^1+\tau_\alpha)\,\gamma_\alpha{}^1\}\,d\Omega=0\tag{241}$$

or the following subsidiary conditions are met, if the functions involved are sufficiently smooth:

$$(L^0_{\alpha\beta\zeta\eta}\chi^1_{\zeta\eta}+\sigma_{\alpha\beta})_{,\beta}-(G^0_{\alpha\beta}\gamma_\beta{}^1+\tau_\alpha)=0\,,\tag{242}$$

$$(G^0_{\alpha\beta}\gamma_\beta{}^1+\tau_\alpha)_{,\alpha}=0\,.\tag{243}$$

One will find that

$$\begin{aligned}
\underline{I}\ (\text{or }\overline{I})=\iint_\Omega\Big\{&U_0(\boldsymbol{\chi}^0,\boldsymbol{\gamma}^0)-\frac{1}{2}\,(\Delta L^{-1}_{\alpha\beta\zeta\eta}\sigma_{\alpha\beta}\sigma_{\zeta\eta}-\sigma_{\alpha\beta}\chi^1_{\alpha\beta}-2\sigma_{\alpha\beta}\chi^0_{\alpha\beta})\\
&-\frac{1}{2}\,(\Delta G^{-1}_{\alpha\beta}\tau_\alpha\tau_\beta-\tau_\alpha\gamma_\alpha{}^1-2\tau_\alpha\gamma_\alpha{}^0)\Big\}\,d\Omega\,.
\end{aligned}\tag{244}$$

We just showed that Eq. (237) has exactly the same structure as the Hashin-Shtrikman-Hill type inequality:

$$\begin{aligned}
\iint_\Omega(U(\boldsymbol{\chi},\boldsymbol{\gamma})-U_0(\boldsymbol{\chi}^0,\boldsymbol{\gamma}^0))\,d\Omega\ \substack{\le\\ \ge}\ -\iint_\Omega\Big\{&\frac{1}{2}\,(\Delta L^{-1}_{\alpha\beta\zeta\eta}\sigma_{\alpha\beta}\sigma_{\zeta\eta}-\sigma_{\alpha\beta}\chi^1_{\alpha\beta}-2\sigma_{\alpha\beta}\chi^0_{\alpha\beta})\\
&+\frac{1}{2}\,(\Delta G^{-1}_{\alpha\beta}\tau_\alpha\tau_\beta-\tau_\alpha\gamma_\alpha{}^1-2\tau_\alpha\gamma_\alpha{}^0)\Big\}\,d\Omega\,.
\end{aligned}\tag{245}$$

The direction of the inequality depends on the positive definiteness or negative definiteness of the tensors $\Delta L^{-1}_{\alpha\beta\zeta\eta}$ and $\Delta G^{-1}_{\alpha\beta}$. Note that for simplicity only two special cases are considered: (i) the tensors $\Delta L^{-1}_{\alpha\beta\zeta\eta}$ and $\Delta G^{-1}_{\alpha\beta}$ are simultaneously positive definite or simultaneously negative definite; (ii) either the moment polarization tensor, $\sigma_{\alpha\beta}$, vanishes, or the shear force polarization vector, τ_α, vanishes.

Similarly, consider the second-type boundary-value problem and introduce two comparison functionals,

$$\Gamma_0(\mathbf{m}^1, \mathbf{Q}^1) \ (\text{or } \Gamma^0(\mathbf{m}^1, \mathbf{Q}^1)) = \frac{1}{2}\iint_\Omega \{N^0_{\alpha\beta\zeta\eta}(m^0_{\alpha\beta} + m^1_{\alpha\beta})(m^0_{\zeta\eta} + m^1_{\zeta\eta}) + H^0_{\alpha\beta}(Q_\alpha{}^0 + Q_\alpha{}^1)(Q_\beta{}^0 + Q_\beta{}^1)\}\, d\Omega\,, \tag{246}$$

if $N_{\alpha\beta\zeta\eta} - N^0_{\alpha\beta\zeta\eta} > 0$ and $H_{\alpha\beta} - H^0_{\alpha\beta} > 0$; or $N_{\alpha\beta\zeta\eta} - N^0_{\alpha\beta\zeta\eta} < 0$ and $H_{\alpha\beta} - H^0_{\alpha\beta} < 0$.

Subsequently, one can form another pair of potential differences:

$$\underline{G}(\mathbf{m}^1, \mathbf{Q}^1) := \Gamma(\mathbf{m}^1, \mathbf{Q}^1) - \Gamma_0(\mathbf{m}^1, \mathbf{Q}^1)\,, \tag{247}$$

$$\overline{G}(\mathbf{m}^1, \mathbf{Q}^1) := \Gamma(\mathbf{m}^1, \mathbf{Q}^1) - \Gamma^0(\mathbf{m}^1, \mathbf{Q}^1)\,. \tag{248}$$

Recall that $(\mathbf{m}^0, \mathbf{Q}^0)$ is the solution of the prescribed force boundary-value problem, and $(\mathbf{m}^1, \mathbf{Q}^1)$ belongs to a closed subspace of $[L^2(\Omega)]^3 \times [L^2(\Omega)]^2$. By defining "polarization curvature" and "polarization rotation" as

$$\eta_{\alpha\beta} := (N_{\alpha\beta\zeta\eta} - N_{\alpha\beta\zeta\eta})\, m_{\zeta\eta}\,, \quad \text{or} \quad \chi_{\alpha\beta} = N^0_{\alpha\beta\zeta\eta} m_{\zeta\eta} + \eta_{\alpha\beta}\,, \tag{249}$$

$$\theta_\alpha := (H_{\alpha\beta} - H^0_{\alpha\beta})\, \gamma_\beta\,, \quad \text{or} \quad \gamma_\alpha = G^0_{\alpha\beta}\gamma_\beta + \theta_\alpha\,, \tag{250}$$

the dual potential differences can be realized as

$$\underline{G^*}(\boldsymbol{\eta}, \boldsymbol{\theta}) = \sup_{(\mathbf{m}^1, \mathbf{Q}^1)\in H} \{(\boldsymbol{\eta}, \mathbf{m}^1) + (\boldsymbol{\theta}, \mathbf{Q}^1) - \underline{G}(\mathbf{m}^1, \mathbf{Q}^1)\}\,, \tag{251}$$

$$\overline{G^*}(\boldsymbol{\eta}, \boldsymbol{\theta}) = \inf_{(\mathbf{m}^1, \mathbf{Q}^1)\in H} \{(\boldsymbol{\eta}, \mathbf{m}^1) + (\boldsymbol{\theta}, \mathbf{Q}^1) - \overline{G}(\mathbf{m}^1, \mathbf{Q}^1)\}\,, \tag{252}$$

which ensures the following estimate of optimization problem (224):

$$\begin{aligned} &\inf_{(\mathbf{m}^1, \mathbf{Q}^1)\in H} \{(\boldsymbol{\eta}, \mathbf{m}^1)(\boldsymbol{\eta}, \mathbf{Q}^1) + \Gamma_0(\mathbf{m}^1, \mathbf{Q}^1)\} - \underline{G^*}(\boldsymbol{\eta}, \boldsymbol{\theta}) \\ &\qquad \le \inf \mathcal{P}_d \le \\ &\inf_{(\mathbf{m}^1, \mathbf{Q}^1)\in H} \{(\boldsymbol{\eta}, \mathbf{m}^1) + (\boldsymbol{\theta}, \mathbf{Q}^1) + \Gamma^0(\mathbf{m}^1, \mathbf{Q}^1)\} - \overline{G^*}(\mathbf{m}^1, \mathbf{Q}^1)\,. \end{aligned} \tag{253}$$

Substituting $m^1_{\alpha\beta} = \Delta N^{-1}_{\alpha\beta\zeta\eta}\eta_{\zeta\eta} - m^0_{\alpha\beta}$ and $Q_\alpha{}^1 = \Delta H^{-1}_{\alpha\beta}\theta_\beta - Q_\alpha{}^0$ into Eqs. (251) and (252), the supremum and infimum will be attained, and it yields

$$\underline{G^*}(\boldsymbol{\eta}, \boldsymbol{\theta}) \ (\text{or } \overline{G^*}(\boldsymbol{\eta}, \boldsymbol{\theta})) = \iint_\Omega \left\{ \frac{1}{2}\Delta N^{-1}_{\alpha\beta\zeta\eta}\eta_{\alpha\beta}\eta_{\zeta\eta} - \eta_{\alpha\beta} m^0_{\alpha\beta} + \frac{1}{2}\Delta H^{-1}_{\alpha\beta}\theta_\alpha\theta_\beta - \theta_\alpha\gamma_\alpha{}^0 \right\} d\Omega\,. \tag{254}$$

Compute

$$\underline{J} := \inf_{(\mathbf{m}^1, \mathbf{Q}^1)\in H} \{(\boldsymbol{\eta}, \mathbf{m}^1) + (\boldsymbol{\theta}, \mathbf{Q}^1) + \Gamma_0(\mathbf{m}^1, \mathbf{Q}^1)\} - \underline{G^*}(\boldsymbol{\eta}, \boldsymbol{\theta})\,, \tag{255}$$

$$\overline{J} := \inf_{(\mathbf{m}^1, \mathbf{Q}^1)\in H} \{(\boldsymbol{\eta}, \mathbf{m}^1) + (\boldsymbol{\theta}, \mathbf{Q}^1) + \Gamma^0(\mathbf{m}^1, \mathbf{Q}^1)\} - \overline{G^*}(\boldsymbol{\eta}, \boldsymbol{\theta})\,. \tag{256}$$

Again recall Remark 3.1; the infima can be reached if the principle of virtual complementary work is applied,

$$\iint_{\Omega} \{(N^0_{\alpha\beta\zeta\eta} m^1_{\zeta\eta} + \eta_{\alpha\beta})\, m^1_{\alpha\beta} + (H^0_{\alpha\beta} Q^1_\beta + \theta_\alpha)\, Q_\alpha{}^1\}\, d\Omega = 0\,, \tag{257}$$

which is equivalent to the following subsidiary conditions if the functions involved are sufficiently smooth:

$$m^1_{\alpha\beta,\beta} - Q_\alpha{}^1 = 0\,, \qquad Q^1_{\alpha,\alpha} = 0\,, \tag{258}$$

$$\varepsilon_{\alpha\eta}(\chi^c_{\alpha\beta,\eta} + \gamma^c_{\eta,\alpha\beta}) + \varepsilon_{\alpha\eta}\chi^c_{\beta\alpha,\eta} = 0\,, \tag{259}$$

where $\chi^c_{\alpha\beta} = N^0_{\alpha\beta\zeta\eta} m^1_{\zeta\eta} + \eta_{\alpha\beta}$ and $\gamma_\alpha{}^c = H^0_{\alpha\beta} Q_\beta{}^1 + \theta_\alpha$.

It is not difficult to find that

$$\begin{aligned} \underline{J}\ (\text{or } \overline{J}) = \iint_{\Omega} \Big\{ & U^*(\mathbf{m}^0, \mathbf{Q}^0) - \frac{1}{2}\,(\Delta N^{-1}_{\alpha\beta\zeta\eta}\eta_{\alpha\beta}\eta_{\zeta\eta} - \eta_{\alpha\beta} m^1_{\alpha\beta} - 2\eta_{\alpha\beta} m^0_{\alpha\beta}) \\ & - \frac{1}{2}\,(\Delta H^{-1}_{\alpha\beta}\theta_\alpha\theta_\beta - \theta_\alpha Q_\alpha{}^1 - 2\theta_\alpha Q_\alpha{}^0) \Big\}\, d\Omega\,. \end{aligned} \tag{260}$$

Consequently, the following variational inequalities hold:

$$\begin{aligned} \iint_{\Omega} (U^*(\mathbf{m}, \mathbf{Q}) - U_0{}^*(\mathbf{m}^0, \mathbf{Q}^0))\, d\Omega \ \begin{matrix} \le \\ \ge \end{matrix}\ & -\frac{1}{2} \iint_{\Omega} \{(\Delta N^{-1}_{\alpha\beta\zeta\eta}\eta_{\alpha\beta}\eta_{\zeta\eta} - \eta_{\alpha\beta} m^1_{\alpha\beta} - 2\eta_{\alpha\beta} m^0_{\alpha\beta}) \\ & + (\Delta H^{-1}_{\alpha\beta}\theta_\alpha\theta_\beta - \theta_\alpha Q_\alpha{}^1 - 2\theta_\alpha Q_\alpha{}^0)\}\, d\Omega\,. \end{aligned} \tag{261}$$

Again, the direction of the inequality is determined by the positiveness or negativeness of the tensors $\Delta N^{-1}_{\alpha\beta\zeta\eta}$ and $\Delta H^{-1}_{\alpha\beta}$.

4 Variation on a theme

To make good use of the comparison variational principles derived above, one needs to know the relationship between the induced deformation field, $(\boldsymbol{\chi}^1, \boldsymbol{\gamma}^1)$, and the resultant polarizations, $(\boldsymbol{\sigma}, \boldsymbol{\tau})$, which is usually in connection with the Eshelby tensor in a circular inclusion problem, if the composite plate is assumed to be macroscopically isotropic. As shown in Sect. 2, for Reissner-Mindlin plates, the deformation field inside an elliptical inclusion is not uniform under the uniform eigen-curvature/eigen-rotation, and the associated Eshelby's tensors have complicate expressions involved with modified Bessel functions, but it turns out that some simple and useful relations between the average induced deformation field and the average polarization field can be obtained for macroscopically isotropic plates under mild restrictions. As a variation on the subject, the following analysis complements the asymptotic results presented in Sect. 2.

4.1 Translation-invariance

We begin with showing that the relationship between the induced deformation field and polarization field is translation-invariant. Consider the auxiliary system (31)–(32) and the subsidiary system,

$$(L^0_{\alpha\beta\zeta\eta}\chi^1_{\zeta\eta} + \sigma_{\alpha\beta})_{,\beta} - (G^0_{\alpha\beta}\gamma_\beta{}^1 + \tau_\alpha) = 0\,, \tag{262}$$

$$(G^0_{\alpha\beta}\gamma_\beta{}^1 + \tau_\alpha)_{,\alpha} = 0\,, \tag{263}$$

with the boundary conditions

$$w^1(\mathbf{x}) = 0\,, \quad (\phi_\alpha{}^1, w^1_{,\alpha}) = 0\,, \quad \Rightarrow \quad \gamma_\alpha{}^1 = \phi_\alpha{}^1 + w^1_{,\alpha} = 0\,, \quad \forall\, \mathbf{x} \in \partial\Omega\,. \tag{264}$$

Multiplying the subsidiary equations (262), (263) with the Green's functions, $(\phi_\alpha{}^{G(k)}, w^{G(k)})$, to form an equation of weighted residual, and by the Gauss theorem one will end with the integration equations

$$\begin{aligned}
\delta_{ik}u_i(\mathbf{x}) = &- \iint_\Omega [\sigma_{\alpha\beta}(\mathbf{x}')\,\chi^{G(k)}_{\alpha\beta}(\mathbf{x},\mathbf{x}') + \tau_\alpha(\mathbf{x}')\,\gamma_\alpha{}^{G(\zeta)}(\mathbf{x},\mathbf{x}')]\,d\Omega' \\
&- \oint_{\partial\Omega} \left(m^{G(k)}_{\alpha\beta}(\mathbf{x},\mathbf{x}')\,n_\beta(\mathbf{x}')\,\phi_\alpha{}^1(\mathbf{x}') + Q_\alpha{}^{G(k)}(\mathbf{x},\mathbf{x}')\,n_\alpha(\mathbf{x}')\,w^1(\mathbf{x}')\right) dS' \\
&+ \oint_{\partial\Omega} [L^0_{\alpha\beta\zeta\eta}\chi^1_{\zeta\eta}(\mathbf{x}') + \sigma_{\alpha\beta}(\mathbf{x}')]\,n_\beta(\mathbf{x}')\,\phi_\alpha{}^{G(k)}(\mathbf{x},\mathbf{x}')\,dS' \\
&+ \oint_{\partial\Omega} [G^0_{\alpha\beta}\gamma^1_{\alpha\beta}(\mathbf{x}') + \tau_\alpha(\mathbf{x}')]\,n_\alpha(\mathbf{x}')\,w^{G(k)}(\mathbf{x},\mathbf{x}')\,dS'\,,
\end{aligned} \tag{265}$$

which can be modified as the following two coupled integration equations:

$$\begin{aligned}
\phi_\zeta(\mathbf{x}) = &- \iint_\Omega \left[\left(\sigma_{\alpha\beta}(\mathbf{x}') - <\sigma_{\alpha\beta}>\right)\chi^{G(\zeta)}_{\alpha\beta}(\mathbf{x},\mathbf{x}') + \left(\tau_\alpha(\mathbf{x}') - <\tau_\alpha>\right)\gamma_\alpha{}^{G(\zeta)}(\mathbf{x},\mathbf{x}')\right] d\Omega' \\
&+ \oint_{\partial\Omega} \left[L^0_{\alpha\beta\zeta\eta}\left(\chi^1_{\zeta\eta}(\mathbf{x}') - <\chi_{\alpha\beta}>\right) + \left(\sigma_{\alpha\beta}(\mathbf{x}') - <\sigma_{\alpha\beta}>\right)\right] n_\beta\phi_\alpha{}^{G(\zeta)}(\mathbf{x},\mathbf{x}')\,dS' \\
&+ \oint_{\partial\Omega} [\tau_\alpha(\mathbf{x}') - <\tau_\alpha>]\,n_\alpha(\mathbf{x}')\,w^{G(\zeta)}(\mathbf{x},\mathbf{x}')\,dS'\,,
\end{aligned} \tag{266}$$

$$\begin{aligned}
w(\mathbf{x}) = &- \iint_\Omega \left[\left(\sigma_{\alpha\beta}(\mathbf{x}') - <\sigma_{\alpha\beta}>\right)\chi^{G(3)}_{\alpha\beta}(\mathbf{x},\mathbf{x}') + \left(\tau_\alpha(\mathbf{x}') - <\tau_\alpha>\right)\gamma_\alpha{}^{G(3)}(\mathbf{x},\mathbf{x}')\right] d\Omega' \\
&+ \oint_{\partial\Omega} \left[L^0_{\alpha\beta\zeta\eta}\left(\chi^1_{\zeta\eta}(\mathbf{x}') - <\chi_{\alpha\beta}>\right) + \left(\sigma_{\alpha\beta} - <\sigma_{\alpha\beta}>\right)\right] n_\beta(\mathbf{x}')\,\phi_\alpha{}^{G(3)}(\mathbf{x},\mathbf{x}')\,dS' \\
&+ \oint_{\partial\Omega} [\tau_\alpha(\mathbf{x}') - <\tau_\alpha>]\,n_\alpha(\mathbf{x}')\,w^{G(3)}(\mathbf{x},\mathbf{x}')\,dS'\,,
\end{aligned} \tag{267}$$

if the boundary conditions in (264) are taken into consideration. One may note that by Lemma 3.1 and its Corollary 3.1

$$\phi_\alpha{}^1 = 0\,, \quad \forall\, \mathbf{x}' \in \partial\Omega \quad \Rightarrow \quad <\chi^1_{\alpha\beta}> = 0\,. \tag{268}$$

Thus, it is plausible that all the boundary terms are oscillating around zero, because the source terms of the transformed moment and transformed resultant oscillate around their means. Based on these arguments, the effects of all the boundary integrals are limited within a boundary layer at the vicinity of the plate's boundary. In other words, for the interior points far away from the boundary, $\partial\Omega$, all the terms in the integrands of (266) and (267) can be neglected.

4.2 Average Eshelby compliance tensors

Being different from both linear elasticity as well as from the Love-Kirchhoff plate, the Reissner-Mindlin plate has two types of polarizations: the moment polarization $\sigma_{\alpha\beta}$ and the shear force polarization τ_α. As can be deduced from (266) and (267), the induced flexural curvature field and the induced shear strain field are coupled in general, i.e.,

$$\boldsymbol{\chi}^1 = -\boldsymbol{\Sigma}^\sigma \boldsymbol{\sigma} - \boldsymbol{\Sigma}^\tau \boldsymbol{\tau}\,, \tag{269}$$

$$\gamma^1 = -\boldsymbol{\Lambda}^\sigma \boldsymbol{\sigma} - \boldsymbol{\Lambda}^\tau \boldsymbol{\tau}\,, \tag{270}$$

where

$$\boldsymbol{\Sigma}^\sigma \boldsymbol{\sigma} = \iint\limits_{\Omega} \boldsymbol{\Sigma}^{(\sigma)\infty}(\mathbf{x}' - \mathbf{x})\,(\boldsymbol{\sigma} - <\boldsymbol{\sigma}>)\,(\mathbf{x}')\,d\Omega'\,, \tag{271}$$

$$\boldsymbol{\Sigma}^\tau \boldsymbol{\tau} = \iint\limits_{\Omega} \boldsymbol{\Sigma}^{(\tau)\infty}(\mathbf{x}' - \mathbf{x})\,(\boldsymbol{\tau} - <\boldsymbol{\tau}>)\,(\mathbf{x}')\,d\Omega'\,, \tag{272}$$

$$\boldsymbol{\Lambda}^\sigma \boldsymbol{\sigma} = \iint\limits_{\Omega} \boldsymbol{\Lambda}^{(\sigma)\infty}(\mathbf{x}' - \mathbf{x})\,(\boldsymbol{\sigma} - <\boldsymbol{\sigma}>)\,(\mathbf{x}')\,d\Omega'\,, \tag{273}$$

$$\boldsymbol{\Lambda}^\tau \boldsymbol{\tau} = \iint\limits_{\Omega} \boldsymbol{\Lambda}^{(\tau)\infty}(\mathbf{x}' - \mathbf{x})\,(\boldsymbol{\tau} - <\boldsymbol{\tau}>)\,(\mathbf{x}')\,d\Omega'\,, \tag{274}$$

and

$$\Sigma^{(\sigma)\infty}_{\zeta\eta\alpha\beta} := -\frac{1}{2}\,(\chi^{G(\zeta)}_{\alpha\beta,\eta} + \chi^{G(\eta)}_{\alpha\beta,\zeta})\,, \tag{275}$$

$$\Sigma^{(\tau)\infty}_{\zeta\eta\alpha} := -\frac{1}{2}\,(\gamma^{G(\zeta)}_{\alpha,\eta} + \gamma^{G(\eta)}_{\alpha,\zeta})\,, \tag{276}$$

$$\Lambda^{(\sigma)\infty}_{\zeta\alpha\beta} := \chi^{G(\zeta)}_{\alpha\beta} - \chi^{G(3)}_{\alpha\beta,\zeta}\,, \tag{277}$$

$$\Lambda^{(\tau)\infty}_{\zeta\alpha} := {\gamma_\alpha}^{G(\zeta)} - \gamma^{G(3)}_{\alpha,\zeta}\,, \tag{278}$$

where

$$\begin{aligned}\chi^{G(\zeta)}_{\alpha\beta} = \frac{\lambda}{2\pi D(1-\nu)} \Bigg\{ \Bigg[& B'(z)\,(\delta_{\zeta\alpha} r_{,\beta} + \delta_{\zeta\beta} r_{,\alpha}) - 2A'(z)\,r_{,\alpha} r_{,\beta} r_{,\zeta} \\ & - \frac{A(z)}{z}\,(\delta_{\zeta\alpha} r_{,\beta} + \delta_{\zeta\beta} r_{,\alpha} + 2\delta_{\alpha\beta} r_{,\zeta} - 4 r_{,\alpha} r_{,\beta} r_{,\zeta}) \Bigg] \\ & - \frac{(1-\nu)}{2z}\,(\delta_{\zeta\alpha} r_{,\beta} + \delta_{\zeta\beta} r_{,\alpha} + \delta_{\alpha\beta} r_{,\zeta} - 2 r_{,\alpha} r_{,\beta} r_{,\zeta}) \Bigg\}\,,\end{aligned} \tag{279}$$

$${\gamma_\alpha}^{G(\zeta)} = \frac{1}{\pi D(1-\nu)}\,[B(z)\,\delta_{\zeta\alpha} - A(z)\,r_{,\zeta} r_{,\alpha}]\,, \tag{280}$$

$$\chi^{G(3)}_{\alpha\beta} = -\frac{1}{8\pi D}\,\big(2 r_{,\alpha} r_{,\beta} + \delta_{\alpha\beta}(2\ln z - 1)\big)\,, \tag{281}$$

$${\gamma_\alpha}^{G(3)} = -\frac{r_{,\alpha}}{\lambda^2 \pi D(1-\nu)\,r}\,, \tag{282}$$

and $r_{,\alpha} = ({x_\alpha}' - x_\alpha)/r, r = |\mathbf{x}' - \mathbf{x}|$.

Introduce the following average Eshelby compliance tensors:

$${\mathbf{P}_R}^\sigma := \iint\limits_{\Omega_R} \boldsymbol{\Sigma}^{(\sigma)\infty}(\mathbf{x})\,d\Omega\,, \tag{283}$$

$${\mathbf{P}_R}^\tau := \iint\limits_{\Omega_R} \boldsymbol{\Sigma}^{(\tau)\infty}(\mathbf{x})\,d\Omega\,, \tag{284}$$

$$\mathbf{R}_R{}^{\sigma} := \iint\limits_{\Omega_R} \Lambda^{(\sigma)\infty}(\mathbf{x})\, d\Omega\,, \tag{285}$$

$$\mathbf{R}_R{}^{\tau} := \iint\limits_{\Omega_R} \Lambda^{(\tau)\infty}(\mathbf{x})\, d\Omega\,, \tag{286}$$

which play a key role on ensuing a variational estimate of effective stiffness of the plates. Note that in this paper we always assume that the distribution of inhomogeneities is macroscopically isotropic and statistically homogeneous. Consequently, the subscript R in Eqs. (283)–(286) indicates that the above average Eshelby compliance tensors are evaluated in a circular inclusion, Ω_R, with a radius R. Next, we evaluate the average Eshelby compliance tensors explicitly.

Remark 4.1. In elastostatics, the tensor **P** has a nice closed form expression, which can be interpreted as the integral of a product of Radon transforms of the Green's function and stress polarization on a unit sphere, if the stress polarization is uniform (see details in Willis [49], and Walpole [45], [46]). □

4.2.1 $P^{\sigma}_{R\,\zeta\eta\alpha\beta}$

$$\begin{aligned}
P^{\sigma}_{R\,\zeta\eta\alpha\beta} : &= -\frac{1}{2}\iint\limits_{\Omega_R} \left(\chi^{G(\zeta)}_{\alpha\beta,\eta}(\mathbf{x}) + \chi^{G(\eta)}_{\alpha\beta,\zeta}(\mathbf{x})\right) d\Omega \\
&= -\frac{1}{2}\oint\limits_{\partial\Omega_R} \left(\chi^{G(\zeta)}_{\alpha\beta}(\mathbf{x})\, n_\eta(\mathbf{x}) + \chi^{G(\eta)}_{\alpha\beta}(\mathbf{x})\, n_\zeta(\mathbf{x})\right) dS \\
&= -\frac{z_R}{4\pi D(1-\nu)}\int_0^{2\pi} \Bigg\{\left(B'(z_R) - \frac{A(z_R)}{z_R} - \frac{(1-\nu)}{2z_R}\right)(\delta_{\zeta\alpha}\ell_\beta\ell_\eta \\
&\quad + \delta_{\zeta\beta}\ell_\alpha\ell_\eta + \delta_{\eta\alpha}\ell_\beta\ell_\zeta + \delta_{\eta\beta}\ell_\alpha\ell_\zeta) - 2\left(\frac{2A(z_R)}{z_R} + \frac{(1-\nu)}{2z_R}\right)\delta_{\alpha\beta}\ell_\zeta\ell_\eta \\
&\quad + 2\left(\frac{4A(z_R)}{z_R} + \frac{(1-\nu)}{z_R} - 2A'(z_R)\right)\ell_\alpha\ell_\beta\ell_\zeta\ell_\eta\Bigg\}\, d\theta \\
&= \frac{1}{8D}\left(f(\nu,\varepsilon_R)\,(\delta_{\zeta\alpha}\delta_{\eta\beta} + \delta_{\zeta\beta}\delta_{\eta\alpha}) + g(\nu,\varepsilon_R)\,\delta_{\alpha\beta}\delta_{\zeta\eta}\right),
\end{aligned} \tag{287}$$

where $\ell_\alpha := x_\alpha/r, \varepsilon_R := \dfrac{R}{(h/\sqrt{12})}$, $z_R = |\varkappa(\nu)\varepsilon_R|$ and

$$f(\nu,\varepsilon_R) = 1 + \frac{2K_1(z_R)\, z_R}{(1-\nu)}\,, \tag{288}$$

$$g(\nu,\varepsilon_R) = 1 + \frac{4A(z_R)}{(1-\nu)} + \frac{2A'(z_R)\, z_R}{(1-\nu)} = 1 - \frac{2K_1(z_R)\, z_R}{(1-\nu)}\,. \tag{289}$$

Remark 4.2.1. There is a universal identity:

$$f(\nu,\varepsilon_R) + g(\nu,\varepsilon_R) = 2\,. \tag{290}$$

2. Unlike the inclusion problem of Love-Kirchhoff plates (Li [26]), the average Eshelby compliance tensor of the Reissner-Mindlin plate depends on the size of the inclusion, the radius of the inclusion, R, to be exact. It is also true that for the ellipsoidal inclusion problem of linear elasticity the Eshelby tensors are independent of the inclusion size (Mura [29]), which leads to the well known Tanaka-Mori Lemma (Tanaka and Mori [39]). Apparently, for Reissner-Mindlin plates, the Eshelby tensors lose their virtue of being independent on the size of the inclusion. On the other hand, this is hardly a vice; such size-dependence automatically brings an intrinsic length sclae into the picture, which oversees the sclae range of the representative-area-element, and in some cases, such as elasto-plastic materials with softening, it can regularize the continuum's governing equations (e.g., Fleck and Hutchinson [10]).[4]

3. The result obtained in (287) has two limits:

(i) $\varepsilon_R \to 0$ as $R \to 0$, in this case

$$f(\nu, 0) = \frac{3-\nu}{1-\nu}, \tag{291}$$

$$g(\nu, 0) = -\frac{1+\nu}{1-\nu}. \tag{292}$$

Moreover, when $\varepsilon_R \ll 1$, it can be found by simple inspection that

$$f(\nu, \varepsilon_R) = f(\nu, 0) + \mathcal{O}({\varepsilon_R}^2), \tag{293}$$

$$g(\nu, \varepsilon_R) = g(\nu, 0) + \mathcal{O}({\varepsilon_R}^2). \tag{294}$$

(ii) $\varepsilon_R \to \infty$ as $h \to 0$, in this case, $z_R K_1(z_R) \to 0$, we find that

$$f(\nu, \infty) = 1, \tag{295}$$

$$g(\nu, \infty) = 1. \tag{296}$$

Therefore, Eq. (287) recovers the result in Love-Kirchhoff plate (see Li [26]). □

4.2.2 $P^{T}_{R\,\zeta\eta\alpha}$

$$\begin{aligned} P^{T}_{R\,\zeta\eta\alpha} := \iint_{\Omega_R} \Sigma^{(\tau)\infty}_{\zeta\eta\alpha}\, d\Omega &= -\frac{1}{2}\iint_{\Omega_R} (\gamma^{G(\zeta)}_{\alpha,\eta} + \gamma^{G(\eta)}_{\alpha,\zeta})\, d\Omega \\ &= -\frac{1}{2}\oint_{\partial\Omega_R} ({\gamma_\alpha}^{G(\zeta)}\eta_\eta + {\gamma_\alpha}^{G(\eta)}\eta_\zeta)\, dS \\ &= \frac{z_R}{2\pi D\lambda(1-\nu)}\int_0^{2\pi} \left\{ \left(B(z_R)\,\delta_{\zeta\alpha} - A(z_R)\,\ell_\zeta \ell_\alpha\right)\ell_\eta \right. \\ &\quad \left. + \left(B(z_R)\delta_{\eta\alpha} - A(z_R)\,\ell_\eta\ell_\alpha\right)\ell_\zeta \right\} d\theta = 0. \end{aligned} \tag{297}$$

Remark 4.3. The main reason that Eq. (297) holds is that the expression in the right-hand side is an odd function of ℓ. Furthermore, it is true that for any smooth function $\omega(|\mathbf{x}|) = \omega(r)$

$$\iint_{\Omega_R} \Sigma^{(\tau)\infty}_{\zeta\eta\alpha}\omega(r)\, d\Omega = 0. \tag{298}$$

[4] In fact, Eshelby tensors are size-dependent in a 3-D Cosserat medium as well (see Cheng and He [3]).

This, too, follows by the fact that the integrand is an odd function of ℓ,

$$\iint_{\Omega_R} \Sigma_{\zeta\eta\alpha}^{(\tau)\infty} \omega(r)\, d\Omega = -\frac{1}{2} \iint_{\Omega_R} (\gamma_{\alpha,\eta}^{G(\zeta)} + \gamma_{\alpha,\zeta}^{G(\eta)})\, \omega(r)\, d\Omega$$
$$= -\frac{1}{2} \oint_{\partial\Omega_R} (\gamma_\alpha{}^{G(\zeta)} n_\eta + \gamma_\alpha{}^{G(\eta)} n_\zeta)\, \omega(R)\, dS$$
$$+\frac{1}{2} \int_0^r \int_0^{2\pi} (\gamma_\alpha{}^{G(\zeta)} \ell_\eta + \gamma_\alpha{}^{G(\eta)} \ell_\zeta)\, \omega'(r)\, r\, dr\, d\theta = 0\,. \qquad \square \quad (299)$$

4.2.3 $R^\sigma_{R\,\zeta\alpha\beta}$

By definition,

$$R^\sigma_{R\,\zeta\alpha\beta} = \iint_{\Omega_R} \Lambda_{\zeta\alpha\beta}^{(\sigma)\infty}(\mathbf{x})\, d\Omega = \iint_{\Omega_R} (\chi_{\alpha\beta}^{G(\zeta)} - \chi_{\alpha\beta,\zeta}^{G(3)})\, d\Omega$$
$$= \oint_{\partial\Omega_R} \left[\frac{1}{2} (\phi_\alpha{}^{G(\zeta)} n_\beta + \phi_\beta{}^{G(\zeta)} n_\alpha) - \chi_{\alpha\beta}^{G(3)} n_\zeta\right] dS$$
$$= \frac{R}{4\pi D} \int_0^{2\pi} \left\{ \frac{2}{1-\nu} \left[B(z_R)\, (\delta_{\zeta\alpha}\ell_\beta + \delta_{\zeta\beta}\ell_\alpha) - 2A(z_R)\, \ell_\alpha \ell_\beta \ell_\zeta\right]\right.$$
$$\left. -\frac{1}{4} (\delta_{\alpha\zeta}\ell_\beta + \delta_{\beta\zeta}\ell_\alpha)\, (2\ln z_R - 1) + \frac{\delta_{\alpha\beta}\ell_\zeta}{2} (2\ln z_R - 1)\right\} d\theta = 0\,. \qquad (300)$$

Remark 4.4 The same statement made in Remark 4.3 is also valid for tensor $R^\sigma_{R\,\zeta\alpha\beta}$. The results (297) and (300) show that the overall couplings between the induced curvature field and shear force polarization, and between the induced rotation field and moment polarization are zero if the composite plate is macroscopically isotropic. $\square$

4.2.4 $R^\tau_{R\,\zeta\alpha}$

Based on (286),

$$R^\sigma_{R\,\zeta\alpha} = \iint_{\Omega_R} \Lambda_{\zeta\alpha}^{(\tau)\infty}(\mathbf{x})\, d\Omega = \iint_{\Omega_R} (\gamma_\alpha{}^{G(\zeta)} - \gamma_{\alpha,\zeta}^{G(3)})\, d\Omega$$
$$= \frac{1}{\lambda^2 \pi D(1-\nu)} \iint_{\Omega_R} [B(z)\, \delta_{\zeta\alpha} - A(z)\, r_{,\zeta} r_{,\alpha}]\, d\Omega + \frac{1}{\lambda^2 \pi D(1-\nu)} \oint_{\partial\Omega} \frac{\ell_\alpha \ell_\zeta}{a}\, dS$$
$$= \frac{\delta_{\zeta\alpha}}{\lambda^2 D(1-\nu)} [1 - z_R K_1(z_R)] + \frac{\delta_{\zeta\alpha}}{\lambda^2 D(1-\nu)} = \frac{\delta_{\zeta\alpha}}{G_p}\, j(\nu, \varepsilon_R)\,, \qquad (301)$$

where $j(\nu, \varepsilon_R) := 1 - \dfrac{z_R K_1(z_R)}{2}$.

Remark 4.5. Function $j(\nu, \varepsilon_R)$, too, has two limits:

(i) $R \to 0 : j(\nu, 0) = 1$ as $\varepsilon_R \to 0$;

(ii) $h \to 0 : j(\nu, \infty) = 1$ as $\varepsilon_R \to \infty$, which leads to

$$\frac{j(\nu,\infty)}{G_p} = \frac{1}{G_p} = \frac{1}{G_p + G_p{}^*}\,. \tag{302}$$

This suggests that $G_p{}^* = 0$ when $\varepsilon_R \to \infty$, which means that there is no additional transformed transverse shear deformation, and the plate behaves as if it were a thin plate. In other words, $\varepsilon_R \to \infty$ characterizes the thin plate limit. □

5 Estimate of overall elastic stiffness

To this end, we are ready to estimate the effective stiffness of composite Reissner-Mindlin plates. Review the isotropic elastic stiffness tensor

$$L_{\alpha\beta\zeta\eta} = \frac{D(1-\nu)}{2}\left(\delta_{\alpha\zeta}\delta_{\beta\eta} + \delta_{\alpha\eta}\delta_{\beta\zeta}\right) + D\nu\delta_{\alpha\beta}\delta_{\zeta}\eta = D(1-\nu)\, I_{\alpha\beta\zeta\eta} + 2D\nu J_{\alpha\beta\zeta\eta}\,, \tag{303}$$

$$G_{\alpha\beta} = G_p\delta_{\alpha\beta} = G_p E_{\alpha\beta}\,, \tag{304}$$

where

$$I_{\alpha\beta\zeta\eta} = \frac{1}{2}\left(\delta_{\alpha\zeta}\delta_{\beta\eta} + \delta_{\alpha\eta}\delta_{\beta\zeta}\right), \tag{305}$$

$$J_{\alpha\beta\zeta\eta} = \frac{1}{2}\,\delta_{\alpha\beta}\delta_{\zeta\eta}\,, \tag{306}$$

$$E_{\alpha\beta} = \delta_{\alpha\beta}\,. \tag{307}$$

Let

$$K_{\alpha\beta\zeta\eta} = \frac{1}{2}\left(\delta_{\alpha\zeta}\delta_{\beta\eta} + \delta_{\alpha\eta}\delta_{\beta\zeta} - \delta_{\alpha\beta}\delta_{\zeta\eta}\right). \tag{308}$$

Thus, $\mathbf{I} = \mathbf{J} + \mathbf{K}$ and

$$\mathbf{J}\cdot\mathbf{J} = \mathbf{J}\,,\quad \mathbf{J}\cdot\mathbf{K} = \mathbf{K}\cdot\mathbf{J} = 0\,,\quad \mathbf{K}\cdot\mathbf{K} = \mathbf{K}\,,\quad \mathbf{E}\cdot\mathbf{E} = \mathbf{E}\,.$$

The elastic stiffness tensor and elastic compliance tensor can then be put into the canonical forms:

$$L_{\alpha\beta\zeta\eta} = 2\varkappa_p J_{\alpha\beta\zeta\eta} + 2\mu_p K_{\alpha\beta\zeta\eta}\,, \tag{309}$$

$$N_{\alpha\beta\zeta\eta} = \frac{1}{2\varkappa_p}\,J_{\alpha\beta\zeta\eta} + \frac{1}{2\mu_p}\,K_{\alpha\beta\zeta\eta}\,, \tag{310}$$

$$G_{\alpha\beta} = G_p E_{\alpha\beta}\,, \tag{311}$$

$$H_{\alpha\beta} = G_p{}^{-1} E_{\alpha\beta}\,, \tag{312}$$

where

$$\varkappa_p := \frac{D(1+\nu)}{2} = \frac{Eh^3}{24(1-\nu)}\,, \tag{313}$$

$$\mu_p := \frac{D(1-\nu)}{2} = \frac{Eh^3}{24(1+\nu)}\,, \tag{314}$$

$$G_p = G\varkappa^2 h \tag{315}$$

or

$$D_p = \varkappa_p + \mu_p \,, \tag{316}$$

$$\nu = \frac{\varkappa_p - \mu_p}{\varkappa_p + \mu_p} \,, \tag{317}$$

$$G_p = D_p \frac{(1-\nu)}{2} \lambda^2 \,. \tag{318}$$

Similarly, the average Eshelby compliance tensors, $\mathbf{P}_R{}^{\sigma}$ and $\mathbf{R}_R{}^{\tau}$ defined by Eqs. (287) and (301), can be also put into the canonical forms:

$$\mathbf{P}_R{}^{\sigma} = \frac{1}{2\varkappa_p{}^0 + 2\varkappa_p{}^*}\,\mathbf{J} + \frac{1}{2\mu_p{}^0 + 2\mu_p{}^*}\,\mathbf{K} \,, \tag{319}$$

$$\mathbf{R}_R{}^{\tau} = \frac{1}{G_p{}^0 + G_p{}^*}\,\mathbf{E} \,, \tag{320}$$

where

$$\varkappa_p{}^* = \mu_p{}^0 \,, \tag{321}$$

$$\mu_p{}^* = \frac{1}{f(\nu, \varepsilon_R)} \left(2\varkappa_p{}^0 + g(\nu, \varepsilon_R)\, \mu_p{}^0\right) , \tag{322}$$

$$G_p{}^* = \left(\frac{1}{j(\nu, \varepsilon_R)} - 1\right) G_p{}^0 \,. \tag{323}$$

5.1 Hashin-Shtrikman type upper/lower bounds

The interesting part of the present formulation is that both polarization quantities, $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$, can excite flexural curvature as well as shear strain. It has been shown in the last section that

$$\begin{aligned}\boldsymbol{\chi}^1 &= -\boldsymbol{\Sigma}^{\sigma}\boldsymbol{\sigma} - \boldsymbol{\Sigma}^{\tau}\boldsymbol{\tau} = -\iint_{\Omega} \boldsymbol{\Sigma}^{(\sigma)\infty}(\mathrm{x}' - \mathrm{x})\,[\boldsymbol{\sigma} - <\boldsymbol{\sigma}>](\mathrm{x}')\, d\Omega' \\ &\quad - \iint_{\Omega} \boldsymbol{\Sigma}^{(\tau)\infty}(\mathrm{x}' - \mathrm{x})\,[\boldsymbol{\tau} - <\boldsymbol{\tau}>]\,(\mathrm{x}')\, d\Omega' \,,\end{aligned} \tag{324}$$

$$\begin{aligned}\boldsymbol{\gamma}^1 &= -\boldsymbol{\Lambda}^{\sigma}\boldsymbol{\sigma} - \boldsymbol{\Lambda}^{\tau}\boldsymbol{\tau} = -\iint_{\Omega} \boldsymbol{\Lambda}^{(\sigma)\infty}(\mathrm{x}' - \mathrm{x})\,[\boldsymbol{\sigma} - <\boldsymbol{\sigma}>](\mathrm{x}')\, d\Omega' \\ &\quad - \iint_{\Omega} \boldsymbol{\Lambda}^{(\tau)\infty}(\mathrm{x}' - \mathrm{x})\,[\boldsymbol{\tau} - <\boldsymbol{\tau}>]\,(\mathrm{x}')\, d\Omega' \,.\end{aligned} \tag{325}$$

Hence, the variational inequality (245) takes the form:

$$\begin{aligned}2(\Pi_0 - \Pi) \lesseqgtr\; & (\boldsymbol{\sigma}, \Delta L^{-1}\boldsymbol{\sigma}) + (\boldsymbol{\tau}, \Delta G^{-1}\boldsymbol{\tau}) + (\boldsymbol{\sigma}, \boldsymbol{\Sigma}^{\sigma}\boldsymbol{\sigma}) + (\boldsymbol{\sigma}, \boldsymbol{\Sigma}^{\tau}\boldsymbol{\tau}) \\ & + (\boldsymbol{\tau}, \boldsymbol{\Lambda}^{\sigma}\boldsymbol{\sigma}) + (\boldsymbol{\tau}, \boldsymbol{\Lambda}^{\tau}\boldsymbol{\tau}) - 2(\boldsymbol{\sigma}, \boldsymbol{\chi}^0) - 2(\boldsymbol{\tau}, \boldsymbol{\gamma}^0) \,.\end{aligned} \tag{326}$$

Following Willis [48], [49], choosing the moment polarization, $\boldsymbol{\sigma}$, and shear force polarization, $\boldsymbol{\tau}$, as piecewise constant distributions and denoting $\boldsymbol{\sigma}^r$ and $\boldsymbol{\tau}^r$ as the values of the polarization fields in rth phase $(r = 1, 2, \cdots, \cdots, n)$, one may convert Eq. (326) to an inequality of the ensemble average,

$$\begin{aligned}
2(\Pi_0 - \Pi) \lesseqgtr\ & \sum_{i=1}^{n} c_i \boldsymbol{\sigma}^i (L_i - L_0)^{-1} \boldsymbol{\sigma}^i + \sum_{i=1}^{n} c_i \boldsymbol{\tau}^i (G_i - G_0)^{-1} \boldsymbol{\tau}^i \\
& + \sum_{i=1}^{n}\sum_{j=1}^{n} \boldsymbol{\sigma}^i \iint_{\Omega} \boldsymbol{\Sigma}^{(\sigma)\infty}(\mathbf{x}) \left[\omega_{ij}(\mathbf{x}) - c_i c_j\right] \boldsymbol{\sigma}^j \, d\Omega \\
& + \sum_{i=1}^{n}\sum_{j=1}^{n} \boldsymbol{\sigma}^i \iint_{\Omega} \boldsymbol{\Sigma}^{(\tau)\infty}(\mathbf{x}) \left[\omega_{ij}(\mathbf{x}) - c_i c_j\right] \boldsymbol{\tau}^j \, d\Omega \\
& + \sum_{i=1}^{n}\sum_{j=1}^{n} \boldsymbol{\tau}^i \iint_{\Omega} \boldsymbol{\Lambda}^{(\sigma)\infty}(\mathbf{x}) \left[\omega_{ij}(\mathbf{x}) - c_i c_j\right] \boldsymbol{\sigma}^j \, d\Omega \\
& + \sum_{i=1}^{n}\sum_{j=1}^{n} \boldsymbol{\tau}^i \iint_{\Omega} \boldsymbol{\Lambda}^{(\tau)\infty}(\mathbf{x}) \left[\omega_{ij}(\mathbf{x}) - c_i c_j\right] \boldsymbol{\tau}^j \, d\Omega \\
& - 2\sum_{i=1}^{n} c_i \boldsymbol{\sigma}^i \bar{\boldsymbol{\chi}} - 2\sum_{j=1}^{n} c_i \boldsymbol{\tau}^i \bar{\boldsymbol{\gamma}} \,.
\end{aligned} \tag{327}$$

Here $\omega_{rs}(\mathbf{x})$ is a special form of the two point correlation function in probability theory, which is chosen as (see Willis [49, p. 35])

$$\omega_{ij}(\mathbf{x}) := P_{ij}(\mathbf{y}, \mathbf{y}+\mathbf{x})\big|_{\mathbf{y}=0}\,, \tag{328}$$

where

$$P_{ij}(\mathbf{y}, \mathbf{y}+\mathbf{x}) := \frac{1}{\Omega}\iint_{\Omega} f_i(\mathbf{y}) f_j(\mathbf{y}+\mathbf{x})\, d\Omega_{\mathbf{y}}\,, \tag{329}$$

and $f(\mathbf{x})$ is the indicator function.

Note that there is a subtle difference between (327) and Eq. (3.4) in Willis [48], and Eq. (327) follows because (see Willis [49, p. 35])

$$\begin{aligned}
& \frac{1}{|\Omega|}\iint_{\Omega} d\mathbf{x}\, f_i(\mathbf{x})\, (\boldsymbol{\Sigma} f_j)\,(\mathbf{x}) \\
& \sim \frac{1}{|\Omega|}\iint_{\Omega} d\mathbf{x}\, f_i(\mathbf{x}) \iint_{\Omega'} d\mathbf{x}' \boldsymbol{\Sigma}^{\infty}(\mathbf{x}' - \mathbf{x})\left(f_j(\mathbf{x}') - c_j\right) \\
& \sim \frac{1}{|\Omega|}\iint_{\Omega} d\mathbf{x}\, f_i(\mathbf{x}) \iint_{\Omega''} d\mathbf{x}'' \boldsymbol{\Sigma}^{\infty}(\mathbf{x}'')\left(f_j(\mathbf{x}+\mathbf{x}'') - c_j\right) \\
& \sim \sum_{i=1}^{n} \iint_{\Omega''} d\mathbf{x}''\, \boldsymbol{\Sigma}^{\infty}(\mathbf{x}'')\left(P_{ij}(\mathbf{x}, \mathbf{x}+\mathbf{x}'') - c_i c_j\right)
\end{aligned}$$

and the assumption that P_{ij} is insensitive to the translation.

For macroscopically isotropic plates, ω_{rs} is assumed to be isotropic, i.e., $\omega(\mathbf{x}) = \omega(|\mathbf{x}|)$. Thereby, as Remark 4.3 suggests,

$$(\boldsymbol{\sigma}, \boldsymbol{\Sigma}^{\tau}\boldsymbol{\tau}) = 0 \quad \text{and} \quad (\boldsymbol{\tau}, \boldsymbol{\Lambda}^{\sigma}\boldsymbol{\sigma}) = 0\,. \tag{330}$$

As mentioned in the Introduction, we are primarily concerned with a composite in which the inhomogeneities have the size scale close to the order of the thickness of the plate. It is, therefore, brutal, but plausible to assume that

$$\omega_{ij}(|\mathbf{x}|) = \begin{cases} c_i c_j, & |\mathbf{x}| > R_c \\ \delta_{ij} c_j, & |\mathbf{x}| \le R_c \end{cases}. \tag{331}$$

Thus,

$$\iint_\Omega \boldsymbol{\Sigma}^{(\sigma)\infty}(\mathbf{x}) \left(\omega_{ij}(|\mathbf{x}|) - c_i c_j\right) d\Omega = \mathbf{P}^\sigma_{R_c}(c_i - c_i c_j)\,, \tag{332}$$

$$\iint_\Omega \boldsymbol{\Lambda}^{(\tau)\infty}(\mathbf{x}) \left(\omega_{ij}(|\mathbf{x}|) - c_i c_j\right) d\Omega = \mathbf{R}^\tau_{R_c}(c_i - c_i c_j)\,. \tag{333}$$

Then, by a standard procedure (again we refer to Willis [48], [49] or Walpole [44], [45]), the following result holds:

$$\bar{\chi}(\mathbf{L} - \bar{\mathbf{L}})\bar{\chi} + \bar{\gamma}(\mathbf{G} - \bar{\mathbf{G}})\bar{\gamma} \ \substack{\le \\ \ge} \ 0\,, \tag{334}$$

where

$$\bar{\mathbf{L}} = \sum_{i=1}^n c_i \mathbf{L}_i \mathbf{A}_i \left(\sum_{j=1}^n c_j \mathbf{A}_j \right)^{-1}, \tag{335}$$

$$\mathbf{A}_i = [\mathbf{I} + \mathbf{P}(\mathbf{L}_i - \mathbf{L}_0)]^{-1}\,, \tag{336}$$

$$\bar{\mathbf{G}} = \sum_{i=1}^n c_i \mathbf{G}_i \mathbf{U}_i \left(\sum_{j=1}^n c_j \mathbf{U}_j \right)^{-1}, \tag{337}$$

$$\mathbf{U}_i = [\mathbf{I} + \mathbf{R}(\mathbf{G}_i - \mathbf{G}_0)]^{-1}\,, \tag{338}$$

where the subscript 0 denotes the properties of the comparison plate.

For a macroscopically isotropic plate,

$$\mathbf{P}^{-1} = \mathbf{L}_0 + \mathbf{L}^*\,, \tag{339}$$

$$\mathbf{R}^{-1} = \mathbf{G}_0 + \mathbf{G}^*\,, \tag{340}$$

and

$$\mathbf{L}^* = 2\varkappa_p{}^* \mathbf{J} + 2\mu_p{}^* \mathbf{K}\,, \tag{341}$$

$$\mathbf{G}^* = G_p{}^* \mathbf{E}\,. \tag{342}$$

In this paper, only a circular inclusion is considered; and we further assume that the inclusions in different phases are in the same length scale. Thus, in the sequel, we simply denote the tensors $\mathbf{P}^\sigma_{R_c}$ and $\mathbf{R}^\tau_{R_c}$ as $\mathbf{P}$ and $\mathbf{R}$, if there is no risk of confusion.

Remark 5.1. 1. A reasonable choice of the critical size of the inclusion would be

$$R_c = \frac{h}{\sqrt{12}}\,, \tag{343}$$

and subsequently

$$P^\sigma_{R_c\,\zeta\eta\alpha\beta} = \frac{1}{8D}\left(f(\nu,1)\left(\delta_{\zeta\alpha}\delta_{\eta\beta} + \delta_{\zeta\beta}\delta_{\eta\alpha}\right) + g(\nu,1)\,\delta_{\alpha\beta}\delta_{\zeta\eta}\right), \tag{344}$$

$$R^\tau_{R_c\,\zeta\alpha} = \frac{\delta_{\zeta\alpha}}{G_p}\,j(\nu,1)\,. \tag{345}$$

The two extreme cases $R_c = 0$ and $R_c \to \infty$ can also be justified under different interpretations: the case $R_c = 0$ is a good approximation for $\varepsilon_{R_c} \ll 1$ and the case $R_c \to \infty$ is a good approximation for $\varepsilon_{R_c} \gg 1$.

2. By using the variational inequalities (261), one can also derive that

$$\bar{\mathbf{m}}(\mathbf{N} - \bar{\mathbf{N}})\,\bar{\mathbf{m}} + \bar{\mathbf{Q}}(\mathbf{H} - \bar{\mathbf{H}})\,\bar{\mathbf{Q}} \ \begin{matrix}\leq \\ \geq\end{matrix}\ 0\,, \tag{346}$$

where

$$\bar{\mathbf{N}} = \sum_{i=1}^{n} c_i \mathbf{N}_r \mathbf{B}_i \left(\sum_{j=1}^{n} c_j \mathbf{B}_j \right)^{-1}, \tag{347}$$

$$\mathbf{B}_i = [\mathbf{I} + \mathbf{O}(\mathbf{N}_i - \mathbf{N}_0)]^{-1}, \tag{348}$$

$$\bar{\mathbf{H}} = \sum_{i=1}^{n} c_i \mathbf{H}_i \mathbf{V}_i \left(\sum_{j=1}^{n} c_j \mathbf{V}_j \right)^{-1}, \tag{349}$$

$$\mathbf{V}_i = [\mathbf{I} + \mathbf{S}(\mathbf{H}_i - \mathbf{H}_0)]^{-1}, \tag{350}$$

with $\mathbf{O} = \mathbf{N}_0 + \mathbf{N}^*$, $\mathbf{N}^* = \dfrac{\mathbf{1}}{\mathbf{2}\varkappa_{\mathbf{p}}{}^*}\,\mathbf{J} + \dfrac{\mathbf{1}}{\mathbf{2}\mu_{\mathbf{p}}{}^*}\,\mathbf{K}$ and $\mathbf{S} = \mathbf{H}_0 + \mathbf{H}^*$, $\mathbf{H}^* = \dfrac{1}{G_p{}^*}\,\mathbf{E}$. □

Let

$$\begin{aligned} \varkappa_g &:= \max_{1\leq i\leq n} \{\varkappa_p{}^{(i)}\} \\ \mu_g &:= \max_{1\leq i\leq n} \{\mu_p{}^{(i)}\} \\ G_g &:= \max_{1\leq i\leq n} \{G_p{}^{(i)}\} \end{aligned} \tag{351}$$

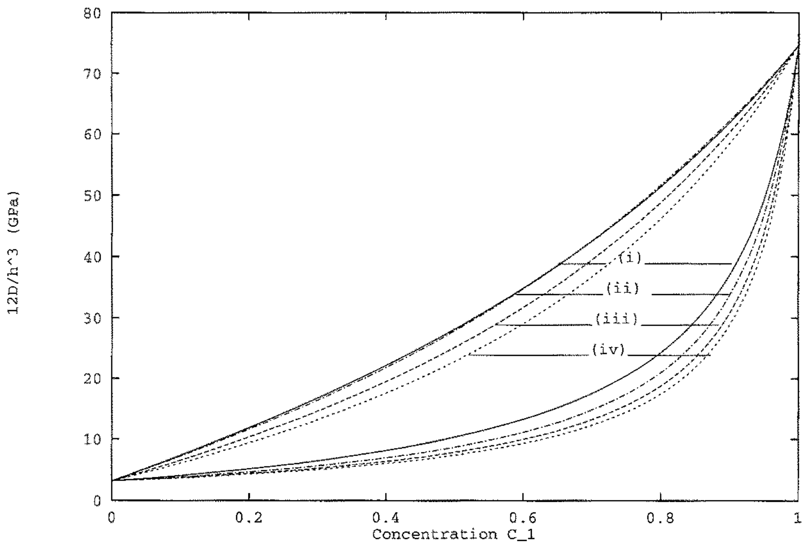
$$\begin{aligned} \varkappa_g{}^* &= \mu_g \\ \mu_g{}^* &= \frac{1}{f(\nu_g, \varepsilon_{R_c})}\left(2\varkappa_g + g(\nu_g, \varepsilon_{R_c})\,\mu_g\right) \\ G_g{}^* &= \left(\frac{1}{j(\nu_g, \varepsilon_{R_c})} - 1\right) G_g\,, \end{aligned} \tag{352}$$

where $\nu_g = (\varkappa_g - \mu_g)/(\varkappa_g + \mu_g)$, and

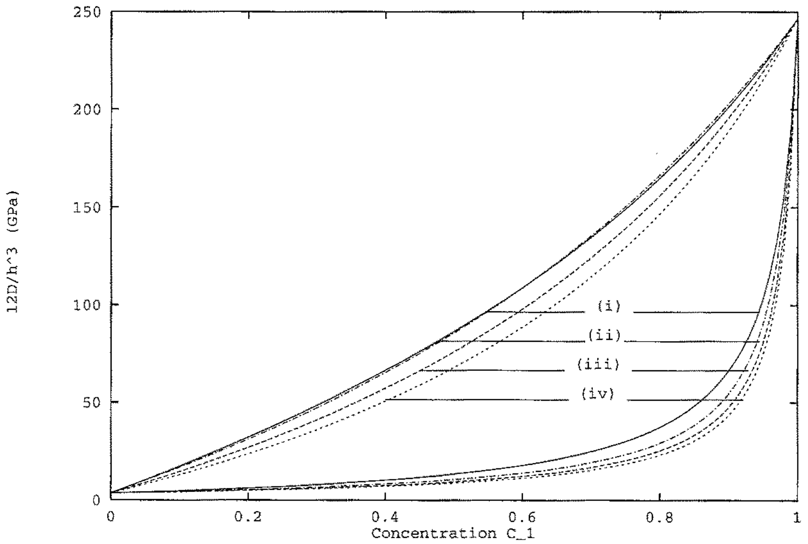
$$\begin{aligned} \varkappa_\ell &:= \min_{1\leq i\leq n} \{\varkappa_p{}^{(i)}\} \\ \mu_\ell &:= \min_{1\leq i\leq n} \{\mu_p{}^{(i)}\} \\ G_\ell &:= \min_{1\leq i\leq n} \{G_p{}^{(i)}\}\,, \end{aligned} \tag{353}$$

$$\begin{aligned} \varkappa_\ell{}^* &= \mu_\ell \\ \mu_\ell{}^* &= \frac{1}{f(\nu_\ell, \varepsilon_{R_c})}\left(2\varkappa_\ell + g(\nu_\ell, \varepsilon_{R_c})\,\mu_\ell\right) \\ G_\ell{}^* &= \left(\frac{1}{j(\nu_\ell, \varepsilon_{R_c})} - 1\right) G_\ell\,, \end{aligned} \tag{354}$$

where $\nu_\ell := (\varkappa_\ell - \mu_\ell)/(\varkappa_\ell + \mu_\ell)$.



a E-glass with epoxy matrix



b Graphite fibers with epoxy matrix

Fig. 6. Congruous bounds on a Reissner-Mindlin plate's rigidity with different R_c: (i) $\varepsilon_{R_c} = \infty$ (the thin plate limit); (ii) Hashin-Shtrikman bound; (iii) $\varepsilon_{R_c} = 1$; (iv) $\varepsilon_{R_c} = 0$

Since (see Walpole [44] for a similar expression)

$$\bar{\mathbf{L}} = \sum_{i=1}^{n} c_i \mathbf{L}_i \mathbf{A}_i \left(\sum_{j=1}^{n} c_j \mathbf{A}_j \right)^{-1} = \left[\sum_{i=1}^{n} c_i [\mathbf{L}_i + \mathbf{L}_0{}^*]^{-1} \right] - \mathbf{L}_0{}^* , \tag{355}$$

we obtain the following estimation on the overall in-plane bulk moduli and shear moduli (on the Cosserat surface):

$$\left[\sum_{i=1}^{n} c_i (\varkappa_\ell{}^* + \varkappa_p{}^{(i)})^{-1} \right] - \varkappa_\ell{}^* \le \varkappa_p \le \left[\sum_{i=1}^{n} c_i (\varkappa_g{}^* + \varkappa_p{}^{(i)})^{-1} \right] - \varkappa_g{}^* , \tag{356}$$

$$\left[\sum_{i=1}^{n} c_i(\mu_\ell{}^* + \mu_p{}^{(i)})^{-1}\right] - \mu_\ell{}^* \le \mu_p \le \left[\sum_{i=1}^{n} c_i(\mu_g{}^* + \mu_p{}^{(i)})^{-1}\right] - \mu_g{}^* , \tag{357}$$

which have the exact same structure or formalism as the classical results of linear elasticity (Hashin and Shtrikman [14], Walpole [44]), but with different physical contents. By adding Eqs. (356) and (357) and utilizing Eq. (316), an explicit estimate for the Reissner-Mindlin plate's rigidity is obtained,

$$\begin{aligned} &\left[\sum_{i=1}^{n} c_i(\varkappa_\ell{}^* + \varkappa_p{}^{(i)})^{-1}\right] + \left[\sum_{i=1}^{n} c_i(\mu_\ell{}^* + \mu_p{}^{(i)})^{-1}\right] - \frac{2D_\ell}{f(\nu_\ell, \varepsilon_{R_c})} , \\ &\qquad\qquad \le D_p \le \\ &\left[\sum_{i=1}^{n} c_i(\varkappa_g{}^* + \varkappa_p{}^{(i)})^{-1}\right] + \left[\sum_{i=1}^{n} c_i(\mu_g{}^* + \mu_p{}^{(i)})^{-1}\right] - \frac{2D_g}{f(\nu_g, \varepsilon_{R_c})} . \end{aligned} \tag{358}$$

Similarly, by considering

$$\bar{\mathbf{G}} = \sum_{i=1}^{n} c_i \mathbf{G}_i \mathbf{U}_i \left(\sum_{j=1}^{n} c_j \mathbf{U}_j\right)^{-1} = \left[\sum_{i=1}^{n} c_i [\mathbf{G}_i + \mathbf{G}_0{}^*]^{-1}\right] - \mathbf{G}_0{}^* , \tag{359}$$

the following estimation on the transverse shear modulus can be obtained:

$$\left[\sum_{i=1}^{n} c_i(G_\ell{}^* + G_p{}^{(i)})^{-1}\right] - G_\ell{}^* \le G_p \le \left[\sum_{i=1}^{n} c_i(G_g{}^* + G_p{}^{(i)})^{-1}\right] - G_g{}^* . \tag{360}$$

Following Hill [18], [19], we define

$$\alpha_{RM} := \frac{\varkappa_p}{\varkappa_p + \varkappa_p{}^*} , \tag{361}$$

$$\beta_{RM} := \frac{\mu_p}{\mu_p + \mu_p{}^*} , \tag{362}$$

$$\varphi_{RM} := \frac{G_p}{G_p + G_p{}^*} , \tag{363}$$

One may verify that

$$\alpha_{RM} = \frac{1+\nu}{2} , \tag{364}$$

$$\beta_{RM} = \frac{f(\nu, \varepsilon_{R_c})\,(1-\nu)}{4} = \frac{1}{4}\left[(1-\nu) + 2|z_R| K_1(z_R)\right] , \tag{365}$$

$$\varphi_{RM} = j(\nu, \varepsilon_{R_c}) = 1 - \frac{z_R K_1(z_R)}{2} . \tag{366}$$

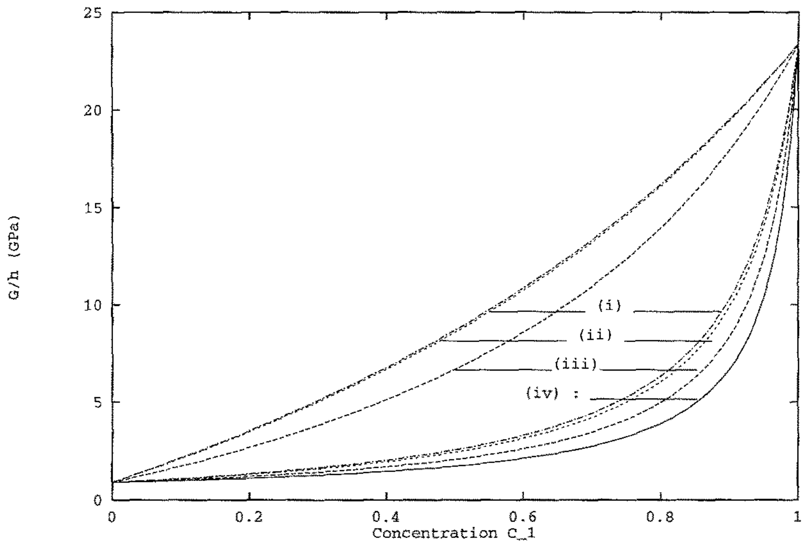
By comparing with their counterparts in both linear elasticity theory (subscript LE) and a Love-Kirchhoff plate theory (subscript LK) (for $-1 < \nu \le 1/2$)

$$\alpha_{LE} = \frac{1+\nu}{3(1-\nu)} , \quad 0 < \alpha_{LE} \le 1 , \tag{367}$$

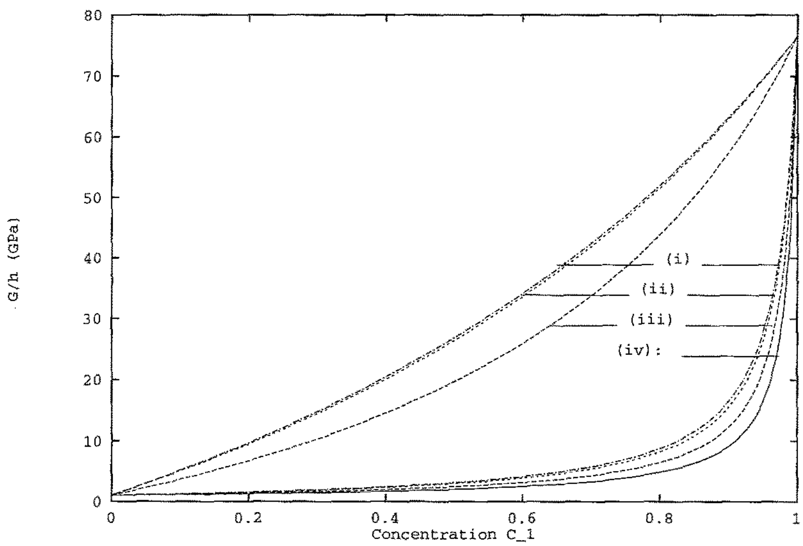
$$\beta_{LE} = \frac{2(4-5\nu)}{15(1-\nu)} , \quad 2/5 \le \beta_{LE} < 3/5 , \tag{368}$$

$$\alpha_{LK} = \frac{1+\nu}{2} , \quad 0 < \alpha_{LK} \le 3/4 , \tag{369}$$

$$\beta_{LK} = \frac{1-\nu}{4} , \quad 1/8 \le \beta_{LK} < 1/2 \tag{370}$$



a E-glass fibers with epoxy matrix



b Graphite fibers with epoxy matrix

Fig. 7. The congruous bounds on the transverse shear modulus with different R_c: (i) Hashin-Shtrikman bound; (ii) $\varepsilon_{R_c} = 0$, (iii) $\varepsilon_{R_c} = 1.0$; (iv) $\varepsilon_{R_c} \to \infty$

the following observations are made:

1. $\alpha_{RM} = \alpha_{LK}\ \forall \varepsilon_{R_c}$, which implies that the variational bounds of a Reissner-Mindlin plate give exact the same estimate on the in-plane bulk modulus as the variational bound of Love-Kirchhoff plate does.

2. As $\varepsilon_R \to \infty$, the variational bounds of the Reissner-Mindlin plate recover the results in a Love-Kirchhoff plate, in other words, in this case, $\beta_{RM} = \beta_{LK}$.

3. The variational bounds of the Reissner-Mindlin plates provide the optimal estimate on the transverse shear modulus, which linear elasticity theory as well as Love-Kirchhoff plate theory do not predict, at least not in an optimal sense.

4. For $\varepsilon_{R_c} = 0$,

$$\alpha_{RM} = \frac{1+\nu}{2}, \quad 0 < \alpha_{RM} \leq 1, \tag{371}$$

$$\beta_{RM} = \frac{3-\nu}{4}, \quad 5/8 \leq \beta_{RM} < 1, \tag{372}$$

$$\varphi_{RM} = \frac{1}{2}. \tag{373}$$

For the thick plates that are made of two phase composite materials, assuming ${\varkappa_p}^{(1)} - {\varkappa_p}^{(2)} > 0$, ${\mu_p}^{(1)} - {\mu_p}^{(2)} > 0$, one will have

$$\varkappa_p{}^{(2)} + \frac{c_1(\varkappa_p{}^{(1)} - \varkappa_p{}^{(2)})}{1 + c_2\alpha_{p2}(\varkappa_p{}^{(1)}/\varkappa_p{}^{(2)} - 1)} \leq \varkappa_p \leq \varkappa_p{}^{(1)} + \frac{c_2(\varkappa_p{}^{(2)} - \varkappa_p{}^{(1)})}{1 + c_1\alpha_{p1}(\varkappa_p{}^{(2)}/\varkappa_p{}^{(1)} - 1)}, \tag{374}$$

$$\mu_p{}^{(2)} + \frac{c_1(\mu_p{}^{(1)} - \mu_p{}^{(2)})}{1 + c_2\beta_{p2}(\mu_p{}^{(1)}/\mu_p{}^{(2)} - 1)} \leq \mu_p \leq \mu_p{}^{(1)} + \frac{c_2(\mu_p{}^{(2)} - \mu_p{}^{(1)})}{1 + c_1\beta_{p1}(\mu_p{}^{(2)}/\mu_p{}^{(1)} - 1)}. \tag{375}$$

They lead to the following estimate of a Reissner-Mindlin plate's rigidity:

$$\begin{aligned} &D_p{}^{(2)} + \frac{c_1(\varkappa_p{}^{(1)} - \varkappa_p{}^{(2)})}{1 + c_2\alpha_{p2}(\varkappa_p{}^{(1)}/\varkappa_p{}^{(2)} - 1)} + \frac{c_1(\mu_p{}^{(1)} - \mu_p{}^{(2)})}{1 + c_2\beta_{p2}(\mu_p{}^{(1)}/\mu_p{}^{(2)} - 1)} \\ &\qquad \leq D_p \leq \\ &D_p{}^{(1)} + \frac{c_2(\varkappa_p{}^{(2)} - \varkappa_p{}^{(1)})}{1 + c_1\alpha_{p1}(\varkappa_p{}^{(2)}/\varkappa_p{}^{(1)} - 1)} + \frac{c_2(\mu_p{}^{(2)} - \mu_p{}^{(1)})}{1 + c_1\beta_{p1}(\mu_p{}^{(2)}/\mu_p{}^{(1)} - 1)}. \end{aligned} \tag{376}$$

Suppose ${G_p}^{(1)} - {G_p}^{(2)} > 0$. The following estimation on the transverse shear moduli is valid:

$$G_p{}^{(2)} + \frac{c_1(G_p{}^{(1)} - G_p{}^{(2)})}{1 + c_2\varphi_{p2}(G_p{}^{(1)}/G_p{}^{(2)} - 1)} \leq G_p \leq G_p{}^{(1)} + \frac{c_2(G_p{}^{(2)} - G_p{}^{(1)})}{1 + c_1\varphi_{p1}(G_p{}^{(2)}/G_p{}^{(1)} - 1)}. \tag{377}$$

Even though it may not be appropriate to use Hashin-Shtrikman bounds in 3-D elasticity in the evaluation of a thick plate's elastic stiffness if the inclusion size is comparable to the thickness of the plate, they, however, provide a set of bounds, at least mathematically. For a two phase composite plate, this could be done by taking the thick plate as a 3-D isotropic elastic medium, and first evaluating the bulk and shear moduli via Hashin-Shtrikman bounds,

$$\lambda_{\text{eff}} = \lambda_{\text{eff}}(\lambda_1, \lambda_2, c_1, c_2), \tag{378}$$

$$G_{\text{eff}} = G_{\text{eff}}(G_1, G_2, c_1, c_2), \tag{379}$$

and then deriving the effective Young's modulus and Poisson's ratio as

$$E_{\text{eff}} = \frac{G_{\text{eff}}(3\lambda_{\text{eff}} + 2G_{\text{eff}})}{\lambda_{\text{eff}} + G_{\text{eff}}}, \tag{380}$$

$$\nu_{\text{eff}} = \frac{\lambda_{\text{eff}}}{2(\lambda_{\text{eff}} + G_{\text{eff}})}, \tag{381}$$

and finally the effective flexural rigidity,

$$D_{\text{eff}} = \frac{E_{\text{eff}} h^3}{12(1 - \nu_{\text{eff}}^2)}. \tag{382}$$

For the two phase composite materials ((a) E-glass fiber and epoxy matrix; and (b) Carbon fiber with epoxy matrix;), we compare the congruous bounds derived in this paper with the bounds according to Hashin-Shtrikman bounds in 3-D elasticity, and plot them in Figs. 6 and 7. In Figs. 6 and 7, the Young's modulus of the E-glass fibers of the first example is 70 GPa, and its Poisson's ratio is 0.25. The Young's modulus of the epoxy matrix in the same example is 2.8 GPa and its Poisson's ratio is 0.35. In the second example, the Young's modulus of the Graphite fibers is 230 GPa, and its Poisson's ratio is 0.26, whereas the Young's modulus of the epoxy matrix in the same example is 3.19 GPa and its Poisson's ratio is 0.35.

From Fig. 6, one may observe that for the flexural rigidity the bounds derived from the Hashin-Shtrikman bounds fall in between the congruous bounds with $\varepsilon_{R_c} = \infty$ and $\varepsilon_{R_c} = 1$. In fact, they are very close to the congruous bounds with $\varepsilon_{R_c} = \infty$, i.e., the thin plate limit. In general, as R_c decreases, the congruous bounds for the flexural rigidity decrease. This indicates that there is a possibility that the Hashin-Shtrikman bounds may overestimate the thick plate's elastic stiffness, if one uses the 3-D elasticity results without discretion. This over-shot tendency becomes obvious when one estimates the plate's transverse shear modulus. Figure 7 presents the comparison between Hashin-Shtrikman bounds and all the congruous bounds. The Hashin-Shtrikman bounds are sitting on top of every other congruous bounds, though very close to the bounds at $\varepsilon_{R_c} = 0$. This makes sense, because when $R_c \to 0$, the inclusion size is tiny, and the plate behaves like an isotropic 3-D continuum, at least macroscopically. Note that, in this case, the upper and lower congruous bounds at $\varepsilon_{R_c} \to \infty$ merge together, lying on the bottom of the series; this happens because there is no transformed rotation at the thin plate limit.

5.2 *Self-consistent estimate*

The similarity between the mathematical structure of the Reissner-Mindlin plate and that of linear elasticity suggests that the conventional self-consistent approximation (Budiansky [1], Hill [18], [19]) for the linear elastic heterogeneous continuum might be valid in a composite Reissner-Mindlin plate as well. However, there are some differences, too. First, a Reissner-Mindlin plate is a Cosserat medium, in which the eigen-curvature not only induces a moment, but also induces a transverse shear force as well; so does the eigen-rotation. Therefore, in general, there are coupling terms between eigen-curvature and eigen-rotation in the overall elastic potential energy. In the following, all the inclusions are assumed to be circular in shape, and the sizes of the inclusions are at the same scale range. In this case, the coupling terms disappear. Second, aloofly speaking, the equivalent inclusion method, Mori-Tanaka method, and some other engineering approaches, etc., all belong to the category of self-consistent methodology in principle, if one disregards the trifled technicalities. In physical principle, they all rely on the fact that the Eshelby tensors are independent on the size of the inclusion, at least in the average sense. This is another place where the generalized eigen-deformation formulation in Reissner-Mindlin plates slightly differs from the classic formulation; consequently, extra care should be taken.

As shown above, when $R_c = 0$ (which corresponds to the first order approximation), the Eshelby tensors, both pointwise and average, are independent of the inclusion size; thus, a self-consistent scheme is a straightforward analogy of the case in 3-D elasticity. A popular choice of the scheme (Hill [18], Budiansky [1]) is the one that assumes that there exist overall constraint stiffness, $\mathbf{L}^*$ and $\mathbf{G}^*$, or compliances, $\mathbf{N}^*$ and $\mathbf{H}^*$, such that in each phase of the plate

$$\bar{\mathbf{m}}_i - \bar{\mathbf{m}} = \mathbf{L}^*(\bar{\chi} - \bar{\chi}_i)\,, \tag{383}$$

$$\bar{\chi}_i - \bar{\chi} = \mathbf{N}^*(\bar{\mathbf{m}} - \bar{\mathbf{m}}_i)\,, \tag{384}$$

$$\bar{\mathbf{Q}}_i - \bar{\mathbf{Q}} = \mathbf{G}^*(\bar{\gamma} - \bar{\gamma}_i)\,, \tag{385}$$

$$\bar{\gamma}_i - \bar{\gamma} = \mathbf{H}^*(\bar{\mathbf{Q}} - \bar{\mathbf{Q}}_i)\,, \tag{386}$$

or

$$(\mathbf{L}^* + \mathbf{L}_i)\,\bar{\chi}_i = (\mathbf{L}^* + \mathbf{L})\,\bar{\chi}\,, \tag{387}$$

$$(\mathbf{N}^* + \mathbf{N}_i)\,\bar{\mathbf{m}}_i = (\mathbf{N}^* + \mathbf{N})\,\bar{\mathbf{m}}\,, \tag{388}$$

$$(\mathbf{G}^* + \mathbf{G}_i)\,\bar{\gamma}_i = (\mathbf{G}^* + \mathbf{G})\,\bar{\gamma}\,, \tag{389}$$

$$(\mathbf{H}^* + \mathbf{H}_i)\,\bar{\mathbf{Q}}_i = (\mathbf{H}^* + \mathbf{H})\,\bar{\mathbf{Q}}\,; \tag{390}$$

here the physical quantities are average moments, average shear resultant, average curvature, and the average rotation. Apparently, the postulate makes sense in the generalized eigenstrain formulation. Substitution of Eqs. (387) and (388) into the identities

$$\sum_{i=1}^{n} c_i(\bar{\mathbf{m}}_i - \bar{\mathbf{m}}) = 0\,, \tag{391}$$

$$\sum_{i=1}^{n} c_i(\bar{\chi}_i - \bar{\chi}) = 0\,, \tag{392}$$

$$\sum_{i=1}^{n} c_i(\bar{\mathbf{Q}}_i - \bar{\mathbf{Q}}) = 0\,, \tag{393}$$

$$\sum_{i=1}^{n} c_i(\bar{\gamma}_i - \bar{\gamma}) = 0\,, \tag{394}$$

yields

$$\sum_{i=1}^{n} \frac{c_i}{\mathbf{L}_i + \mathbf{L}} = \frac{1}{\mathbf{L}^* + \mathbf{L}} = \mathbf{P} \quad \Rightarrow \quad \sum_{i=1}^{n} c_i[(\mathbf{L}_i - \mathbf{L})^{-1} + \mathbf{P}]^{-1} = 0\,, \tag{395}$$

$$\sum_{i=1}^{n} \frac{c_i}{\mathbf{N}_i + \mathbf{N}} = \frac{1}{\mathbf{N}^* + \mathbf{N}} = \mathbf{O} \quad \Rightarrow \quad \sum_{i=1}^{n} c_i[(\mathbf{N}_i - \mathbf{N})^{-1} + \mathbf{O}]^{-1} = 0\,, \tag{396}$$

$$\sum_{i=1}^{n} \frac{c_i}{\mathbf{G}_i + \mathbf{G}} = \frac{1}{\mathbf{G}^* + \mathbf{G}} = \mathbf{S} \quad \Rightarrow \quad \sum_{i=1}^{n} c_i[(\mathbf{G}_i - \mathbf{G})^{-1} + \mathbf{S}]^{-1} = 0\,, \tag{397}$$

$$\sum_{i=1}^{n} \frac{c_i}{\mathbf{H}_i + \mathbf{H}} = \frac{1}{\mathbf{H}^* + \mathbf{H}} = \mathbf{T} \quad \Rightarrow \quad \sum_{i=1}^{n} c_i[(\mathbf{H}_i - \mathbf{H})^{-1} + \mathbf{S}]^{-1} = 0\,, \tag{398}$$

or in equivalent forms

$$\mathbf{L} = \left\{ \sum_{i=1}^{n} c_i \mathbf{L}_i[\mathbf{I} + \mathbf{P}(\mathbf{L}_i - \mathbf{L})]^{-1} \right\} \left\{ \sum_{j=1}^{n} c_j[\mathbf{I} + \mathbf{P}(\mathbf{L}_j - \mathbf{L})]^{-1} \right\}^{-1}, \tag{399}$$

$$\mathbf{N} = \left\{ \sum_{i=1}^{n} c_i \mathbf{N}_i[\mathbf{I} + \mathbf{O}(\mathbf{N}_i - \mathbf{N})]^{-1} \right\} \left\{ \sum_{j=1}^{n} c_j[\mathbf{I} + \mathbf{O}(\mathbf{N}_j - \mathbf{N})]^{-1} \right\}^{-1}, \tag{400}$$

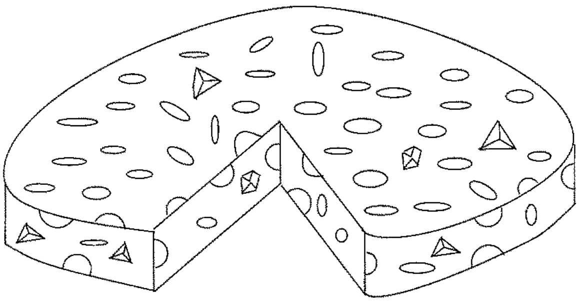


Fig. 8. A piece of "Swiss cheese" – a thick plate with distributed cavities

$$\mathbf{G}=\left\{\sum_{i=1}^{n}c_i\mathbf{L}_i[\mathbf{I}+\mathbf{R}(\mathbf{G}_i-\mathbf{G})]^{-1}\right\}\left\{\sum_{j=1}^{n}c_j[\mathbf{I}+\mathbf{R}(\mathbf{G}_j-\mathbf{G})]^{-1}\right\}^{-1}, \tag{401}$$

$$\mathbf{H}=\left\{\sum_{i=1}^{n}c_i\mathbf{H}_i[\mathbf{I}+\mathbf{S}(\mathbf{H}_i-\mathbf{H})]^{-1}\right\}\left\{\sum_{j=1}^{n}c_j[\mathbf{I}+\mathbf{S}(\mathbf{H}_j-\mathbf{H})]^{-1}\right\}^{-1}, \tag{402}$$

To entertain the thought that this self-consistent scheme for the Reissner-Mindlin plate is practically useful in what follows, we calculate the elastic stiffness for a special two-phase composite plate – a sheet of "Swiss cheese" (Fig. 8), in other words, one phase of the composite (say phase 1) is taken as cavity, which implies that $\mathbf{L}^{(1)}=0$ or ${\varkappa_p}^{(1)}={\mu_p}^{(1)}=0$.

For the two phase composite plate, Eq. (395) take the form

$$\frac{c_1}{\mathbf{L}-\mathbf{L}_2}+\frac{c_2}{\mathbf{L}-\mathbf{L}_1}=\mathbf{P}, \tag{403}$$

where ($\varepsilon_{r_c}=0$)

$$\mathbf{P}=\left(\frac{\alpha_p}{2\varkappa_p},\frac{\beta_p}{2\mu_p}\right)\quad\text{and}\quad\alpha_p=\frac{1+\nu}{2},\quad\beta_p=\frac{3-\nu}{4}. \tag{404}$$

Let ${\varkappa_p}^{(1)}={\mu_p}^{(1)}=0$. We can solve $\varkappa_p$ in terms of μ_p, i.e.,

$$\varkappa_p=\frac{c_2{\varkappa_p}^{(2)}\mu_p}{c_1{\varkappa_p}^{(2)}+\mu_p}, \tag{405}$$

and μ_p, D_p can be solved explicitly without further assumption on ${\varkappa_p}^{(2)}$[5],

$$\mu_p=\frac{{\varkappa_p}^{(2)}{\mu_p}^{(2)}(1-3c_1)}{2c_1{\mu_p}^{(2)}+c_2{\varkappa_p}^{(2)}}, \tag{406}$$

$$D_p=\frac{{D_p}^{(2)}{\varkappa_p}^{(2)}{\mu_p}^{(2)}(1-3c_1)}{(2c_1{\mu_p}^{(2)}+c_2{\varkappa_p}^{(2)})[(2c_2-1)\,{\mu_p}^{(2)}+c_1{\varkappa_p}^{(2)}]}. \tag{407}$$

Obviously, the volume of the cavity should not exceed one third of the total volume of the plate. It is interesting to compare this with the similar problems solved in the context of 3-D elasticity (see Willis [49]) and a thin plate. In 3-D elasticity, the total volume of the cavity

[5] The same problem in linear elasticity as well as in thin plate theory are treated under the assumption that the matrix is incompressible, i.e., ${\varkappa_p}^{(2)}\to\infty$.

should not exceed one half of the total volume of the medium, whereas in a thin plate (Li [26]) the total volume of the cavity should not exceed two third of the total volume of the plate.

However, the interesting part has not ended yet. Let is consider the transverse shear modulus for $\varepsilon_{R_c} = 0$, which amend $G_p{}^* = G_p$. From the self-consistent scheme (397), one will have the equation

$$\frac{c_1}{G_p - G_p{}^{(2)}} + \frac{c_2}{G_p - G_p{}^{(1)}} = \frac{c_1}{G_p{}^{(1)} + G_p} + \frac{c_2}{G_p{}^{(2)} + G_p}\,, \tag{408}$$

which yields

$$G_p = \frac{1}{2}\left\{(c_1 - c_2)\,(G_p{}^{(1)} - G_p{}^{(2)}) + \sqrt{(c_1 - c_2)^2\,(G_p{}^{(1)} - G_p{}^{(2)})^2 + 4G_p{}^{(1)}G_p{}^{(2)}}\right\}. \tag{409}$$

If $G_p{}^{(2)} = 0$, we end up with

$$G_p = (1 - 2c_1)\,G_p{}^{(2)}\,. \tag{410}$$

Again, the total volume of the cavity should not exceed one half of the total volume of the composite plate, and this corresponds to the elasticity result.

6 Concluding remarks

Even though the methodology of micromechanics has been extensively used in the study of composite materials and has become an indispensable part of composite mechanics, in current engineering practice the design criteria as well as the standards in the strength analysis of composite structures are still limited within the realm of conventional laminar plate theories or laminar shell theories.

To address this inadequacy, this work presents a systematic study on the micromechanics that is congruous with the Reissner-Mindlin plate theory, which can be applied to the cases that thick plates are made by embedding short fibers, or functional cells, which provide the reinforcement to structures. The present formulation is attractive because, first, it preserves all the original assumptions of the Reissner-Mindlin plate theory and hence the validity and generality in applications, and second, it preserves the rigor of the micro-elasticity, hence the elegance and permanence in its theoretical value.

The main contribution of this work is in the following three aspects: (i) analytical solutions on the elliptical inclusion problem of the Reissner-Mindlin plate, (ii) comparison variational principles for the Reissner-Mindlin plate, and (iii) the congruous bounds on the elastic stiffness of the plate.

As an analogy of micromechanics in linear elasticity theory, the micromechanics of a Reissner-Mindlin plate may be further generalized to the general 2-D Cosserat-manifold, i.e. the elastic shells that are capable to sustain shear deformation. Because of the complexity of the corresponding Green's function, one might expect to deal with them on a specific basis. In the end, we would like to note that even though the results given in this paper resemble the classic formalism in character, they obviously differ in quantities, and, most importantly, in the physical implications. Nevertheless, the applications may be subjected to certain restraints, such as the distribution patterns of the inhomogeneities, and the shape of the inclusions.

Appendix:

Evaluation of tensor $S^{FM}_{\alpha\beta\zeta\eta}$

Based on the formula (98), the components of tensor $S^{FM}_{\alpha\beta\zeta\eta}$ are calculated as follows:

$$S_{1111} = \frac{(1-\nu)}{4\pi} I_1 + \frac{(1+\nu)\,a_1{}^2}{2\pi} I_{11}\,, \tag{A.1}$$

$$S_{1122} = \frac{(1-\nu)}{4\pi} I_1 + \frac{(1+\nu)\,a_2{}^2}{2\pi} I_{12}\,, \tag{A.2}$$

$$S_{1212} = S_{2121} = \frac{(1-\nu)}{8\pi} (I_1 + I_2) + \frac{(1+\nu)}{4\pi} (a_1{}^2 + a_2{}^2)\, I_{12}\,, \tag{A.3}$$

$$S_{2211} = \frac{(1+\nu)}{4\pi} I_2 + \frac{(1+\nu)\,a_1{}^2}{2\pi} I_{21}\,, \tag{A.4}$$

$$S_{2222} = \frac{(1-\nu)}{4\pi} I_2 + \frac{(1+\nu)\,a_2{}^2}{2\pi} I_{22}\,, \tag{A.5}$$

$$S_{1112} = S_{1121} = 0\,, \tag{A.6}$$

$$S_{2212} = S_{2221} = 0\,, \tag{A.7}$$

$$S_{1211} = S_{2111} = 0\,, \tag{A.8}$$

$$S_{1222} = S_{2122} = 0\,, \tag{A.9}$$

where

$$I_\mu = \frac{1}{a_\mu{}^2} \int_0^{2\pi} \frac{\ell_\mu{}^2}{g}\, d\theta\,, \qquad \mu = 1, 2\,, \tag{A.10}$$

$$I_{\alpha\beta} = \frac{1}{a_\alpha{}^2 a_\beta{}^2} \int_0^{2\pi} \frac{\ell_\alpha{}^2 \ell_\beta{}^2}{g}\, d\theta\,, \qquad \alpha, \beta = 1, 2\,, \tag{A.11}$$

which satisfy the following identities:

$$I_1 + I_2 = 2\pi\,, \tag{A.12}$$

$$a_1{}^2 I_{11} + (a_1{}^2 + a_2{}^2)\, I_{12} + a_2{}^2 I_{22} = 2\pi\,, \tag{A.13}$$

$$\frac{a_1{}^2}{a_2{}^2} I_{11} + 2I_{12} + \frac{a_2{}^2}{a_1{}^2} I_{22} = \frac{1}{a_2{}^2} I_1 + \frac{1}{a_1{}^2} I_2\,. \tag{A.14}$$

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